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Geometries and solutions of some steady magneto fluid dynamic flows.

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GEOMETRIES AND SOLUTIONS OF SOME STEADY MAGNETO FLUID DYNAMIC FLOWS

by

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ABSTRACT

Steady MFD flows of fluids with infinite electrical conductivity are studied for the following cases:

(a) Inviscid, Aligned Flows.

(b) Inviscid, Transverse Flows.

(c) Incompressible, Non-Aligned Flows.

A substitution principle is developed for inviscid aligned and transverse flows when the equation of state of the fluid is of an arbitrary form. Geometries of the flows, for which the substitution principle holds, are obtained.

For incompressible, inviscid aligned, transverse and orthogonal flows, the conditions for two dynamically distinct flows to have the same streamline pattern are determined.

For inviscid aligned flows geometries of plane, axially-symmetric, spatial irrotational and spatial doubly-laminar flows, when velocity magnitude is constant along each streamline, are obtained.

For inviscid transverse flows a linear hyperbolic partial differential equation of second order in speed is obtained by eliminating other flow variables from the flow equations in natural co-ordinates. The speed equation is solved for flow through a logarithmic channel.

For inviscid orthogonal flows, geometries of the flows are obtained when either the local speed of sound, or the pressure or the density or the speed of the fluid is constant along each streamline. The conditions under which the pressure is constant along orthogonal trajectories in an isometric net are found. The geometries of straight streamlines for fluids having product equation of state, are studied.

Finally, for plane, incompressible, non-aligned flows, a new form of the fundamental equations is obtained. These newly formed equations are used to prove the following:

(i) For viscid, orthogonal flows, if the streamlines are straight lines but not parallel, then they must be concurrent.

(ii) For viscid, orthogonal flows, if the streamlines are involutes of a curve, then the streamlines are concentric circles.

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CHAPTER I

INTRODUCTION

(A) Historical Sketch.

Magneto-Fluid Dynamics is the study of flow of an electrically conducting fluid when radiation field is absent and the energy in the electric field is much smaller than that in the magnetic field. This subject is governed by a system of non-linear partial differential equations arising from conservation laws. In addition to the presence of the thermodynamic variables, the interaction between the magnetic field and the gas dynamic field present in these equations has provided quite a challenge. S. Lunquist (1952) studied unsteady flows with infinite electrical conductivity. Some developments in the field of linearized Magneto-Fluid Dynamics have been made by W.R. Sears (1959), E.L. Resler (1959) and their group for inviscid fluids. A diversity of problems in MFD have been handled by, among others, J.A. Shercliff (1953, 1956), S. Chandrasekhar (1961), Shih-I Pai (1962) and K.B. Ranger (1969). The complexity of this subject is well understood when one finds that many of the flow problems, even when linearized, are abstruse and difficult to comprehend. To overcome this complication and to make progress in this relatively new subject of dynamics, the researchers are using an alternative technique of first comprehending and analysing special classes of flows. This approach helps to study the similarities and the contrasts with conventional fluid dynamical flows. Furthermore, this procedure serves to separate flow problems which are relatively accessible using fluid dynamical techniques from the remainder which may require the development of new techniques.

In the following the different types of flows are considered in relation to the work existing on gas flows and the current status of the work on MFD.

Aligned Flows.

Flows are said to be parallel or aligned if the streamlines and the magnetic lines are coincident.

Aligned MFD flows may be studied by extensions of methods used to study rotational gas flows. The fundamental work in steady rotational flow of ideal gases can be attributed to R.C. Prim (1952). Prim put forward a substitution principle for steady, inviscid, non-conducting fluids.

H. Grad (1960), P. Germain (1960), M. Vinokur (1961), R. Peyret (1960) developed many fundamental results for aligned flows when specific entropy is taken constant along each streamline. Bernoulli equation and kinematic formulations studied by these authors led P. Smith (1963) to generalize some of R.C. Prim's discoveries in steady rotational flows of ideal gases. Smith extended the Prim's substitution principle to steady, inviscid fluids with infinite electrical conductivity in a rough and non-elegant way. This principle was derived for fluids with the equation of state in product form. Smith also derived the canonical equations. Finally, he studied radial, vortex and Ringleb's plane flows of polytropic gases with uniform stagnation pressure.

Other works that use different approaches to solving aligned flow problems have been referred to by A. Jeffrey (1966) in Chapter VII of his book.

Transverse Flows.

Flows are said to be transverse if the magnetic field is normal to the plane of flow and all the flow variables including the magnetic field vector depend upon planar co-ordinates only.

H. Grad (1960) derived two independent integrals for transverse flows, one relating the magnetic induction with the speed of sound and the other a generalized Bernoulli relation. P. Smith (1963), using Prim's substitution principle considered transverse flows to a rather limited extent. R.M. Gundersen (1966) studied simple wave flows and their characteristics by using the two integrals derived by Grad when the flow is irrotational and homentropic. On these flows, this author published two more papers in 1966 and 1969. In the first paper the geometric condition so that the velocity magnitude is constant along a streamline was studied. This result was proved for polytropic gases. In his second paper he used M.H. Martin's (1950) and D. Naylor's (1954) approach to study transverse flows.

Orthogonal Flows.

Flows are said to be orthogonal if the velocity field vector and the magnetic field vector are everywhere orthogonal.

Iu.P. Ladikov (1962) derived two integrals, of which one is analogous to Bernoulli integral and the other does not have any correspondence in ordinary gas dynamics, for these flows. This author also studied homentropic, radial and vortex flows of polytropic gases.

G. Power and his co-workers (1965, 1967, 1969) studied the reduction of viscous flows to flows with zero magnetic field, linked inviscid, plane flows to four parameter class of compressible fluid flows with zero magnetic field and determined the geometry of flow when the velocity magnitude is constant along each individual streamline. However, the last result was proved by these authors by employing some additional assumptions.

(B) Scope of the Present Work.

It is apparent from the survey of the literature that studies in aligned flows, transverse flows or orthogonal flows are not complete. The present work is intended to answer some more questions on geometries and solutions of these flows.

The basic approach used in the present work is to develop solutions for MFD flows on the basis of techniques that have been found successful for non-conducting fluid flows. In this aspect the present work uses the same philosophy as many of the works referred to earlier. The difference is, of course, that completely new problems have been considered, particularly in terms of geometry, and also that new solution procedures have been developed where necessary. The works in fluid flow of primary importance for the present work are those by D. Gilbarg (1947), R.C. Prim (1952), J.L. Ericksen (1952), O.P. Chandna and A.C. Smith (1971).

The problems in MFD flows that have been considered in this thesis are:

(i) Aligned Flows. Development of substitution principle; plane and three dimensional flows. In three dimensional flows, axially-symmetric, spatial irrotational and doubly-laminar flows are considered.

(ii) Transverse Flows. Development of substitution principle; speed equation; logarithmic channel flows.

(iii) Orthogonal Flows. Development of geometric theorems; source flows; vortex flows; straight parallel flows.

(iv) Viscous, non-aligned, incompressible flows. Generalization based on M.H. Martin's (1971) work.

In the next section, an outline of the work presented in this thesis is given.

(C) <u>Outline of the Present Work</u>.

In section 1 of chapter II we write the equations of motion of Magneto-Fluid Dynamics in their most general form. In section 2 of this chapter we give some results of differential geometry to be used later.

Chapter III deals with aligned flows of inviscid, thermally non-conducting, electrically infinitely conducting fluids. Here, we study the following:

(i) The physical and the geometrical restrictions so that the substitution principle holds for fluids having arbitrary equation of state.

(ii) Restrictions for the flows to be sonic and also conditions that flows have concentric circles or parallel straight lines as the streamlines.

(iii) Physical and geometrical conditions so that the set of all dynamically distinct incompressible flows have the same streamline pattern as that of the given flow.

(iv) The technique of reducing the problem of plane, compressible, homentropic, irrotational, aligned flows to that of plane, compressible, homentropic, irrotational flows of non-conducting gases.

(v) Plane source flows and find the general solution.

In Chapter IV we study transverse flows of inviscid, thermally non-conducting and electrically infinitely conducting fluids. In case of these flows we find that

(i) The substitution principle for flows with arbitrary equation of state holds provided one of the scalar quantities the velocity magnitude V or the pressure p or the density ρ or the local speed of sound C is constant along each streamline or the speed is sonic. These flows are either vortex flows or flows in parallel straight lines.

(ii) Any flow is unique unless the velocity magnitude is constant along each streamline.

In section 5 we find a linear hyperbolic partial differential equation of second order in speed V, from the flow equations in natural co-ordinates, by eliminating the other flow variables. In section 6 we find the density, the pressure, the specific entropy and the magnetic field vector in terms of the velocity magnitude. In sections seven, eight and nine we obtain solution to flows in a logarithmic channel.

In chapter V, we study the plane Magneto-Fluid Dynamic flows with orthogonal magnetic and velocity field distributions. In section 1 of this chapter the non-linear partial differential equations governing the flow of inviscid, thermally non-conducting and electrically infinitely conducting fluids are formulated in the curvilinear co-ordinate system (ϕ,ψ) where ψ = constant are the streamlines and ϕ = constant are the magnetic lines.

In section 2 we investigate the following:

(i) The conditions under which the streamlines coincide with lines of constant velocity magnitude for fluids with arbitrary equation of state.

(ii) The conditions under which the streamlines coincide with lines of sonic velocity magnitude for fluids with arbitrary equation of state.

(iii) The conditions on pressure, density, speed of sound, velocity magnitude and vorticity if any one of the first three variables is constant along the streamlines.

(iv) The conditions under which the orthogonal trajectories are isobaric curves in an orthogonal isometric net.

(v) The conditions on straight streamlines for fluids having equation of state in the product form.

In section 3 we prove that if ψ^* and ψ are the stream functions of two dynamically distinct flows having the same streamline pattern, then $\psi^* = A\psi + B$ where A and B are arbitrary constants.

In section 4 we find general solutions to the problems of straight parallel, vortex and source flows.

Following Martin's (1971) approach, in chapter VI, we show that when streamlines ψ = constant and magnetic lines ϕ = constant of plane, non-aligned flow of a viscous, incompressible fluid of infinite electrical conductivity are taken as the curvilinear co-ordinate system the fundamental equations governing the flow can be replaced by a new system of equations. In these equations ϕ, ψ are the independent variables.

In case of orthogonal flows we prove the following:

(i) If the streamlines are straight lines but not parallel, then they must be concurrent.

(ii) If the streamlines are involutes of a curve, then the streamlines are concentric circles.

Finally, we find solutions to vortex and source flow problems.

CHAPTER II

PRELIMINARIES

Section 1. General Equations of Motion of Magneto-Fluid Dynamics (MFD).

The fundamental equations of MFD governing the flow of thermally non-conducting and electrically conducting fluids are:

$$\frac{\partial \rho}{\partial t} + \text{div } \rho \vec{V} = 0 \qquad (21.01)$$

(Conservation of Mass)

$$\begin{aligned}\rho \frac{\partial \vec{V}}{\partial t} + \rho (\vec{V} \cdot \text{grad}) \vec{V} + \text{grad } p \\ = \frac{4}{3} \text{grad } (\eta \text{ div } \vec{V}) + \text{grad } (\vec{V} \cdot \text{grad } \eta) \\ - \vec{V} \text{ div } (\text{grad } \eta) - \text{curl } (\text{curl } \eta \vec{V}) \\ + \text{grad } \eta \times \text{curl } \vec{V} - (\text{div } \vec{V}) \text{ grad } \eta \\ + \mu (\text{curl } \vec{H}) \times \vec{H}\end{aligned} \qquad (21.02)$$

(Conservation of Momentum)

$$\begin{aligned}\rho T \frac{\partial s}{\partial t} + \rho T \vec{V} \cdot \text{grad } s \\ = \eta \sum_{ij=1}^{3} \left[\left\{\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \text{ div } \vec{V}\right\} \frac{\partial v_i}{\partial x_j}\right] \\ + \frac{1}{\sigma} (\text{curl } \vec{H})^2 + \text{div } (\kappa \text{ grad } T)\end{aligned} \qquad (21.03)$$

(Energy Equation)

$$\rho = \rho(p,s) \quad \text{or} \quad \rho = \rho(p,T) \qquad (21.04)$$

(Equation of State)

$$\text{curl } (\vec{V} \times \vec{H}) - \text{curl} \left(\frac{1}{\mu\sigma} \text{ curl } \vec{H}\right) = \frac{\partial \vec{H}}{\partial t} \qquad (21.05)$$

(Equation for Magnetic Field)

where $\vec{V}$ denotes the velocity vector, ρ the density, p the pressure, s the specific entropy, η the coefficient of viscosity, μ the magnetic permeability, $\vec{H}$ the solenoidal magnetic field vector, σ the electrical conductivity, κ the thermal conductivity, T the absolute temperature of the fluid and δ_{ij} is the Kronecker delta.

Section 2. Some Results from Differential Geometry.

Let

$$x = x(\phi,\psi) \quad , \quad y = y(\phi,\psi) \qquad (22.01)$$

define a system of curvilinear co-ordinates in the (x,y) - plane. In the curvilinear co-ordinate system (ϕ,ψ) the squared element of arc length is given by

$$ds^2 = E\, d\phi^2 + 2F\, d\phi\, d\psi + G\, d\psi^2 \qquad (22.02)$$

where

$$\left.\begin{aligned} E &= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 \\ F &= \left(\frac{\partial x}{\partial \phi}\right)\left(\frac{\partial x}{\partial \psi}\right) + \left(\frac{\partial y}{\partial \phi}\right)\left(\frac{\partial y}{\partial \psi}\right) \\ G &= \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 \end{aligned}\right\} \qquad (22.03)$$

Equations (22.01) can be used to obtain

$$\phi = \phi(x,y)$$

$$\psi = \psi(x,y)$$

such that

$$\left(\frac{\partial x}{\partial \phi}\right) = J\left(\frac{\partial \psi}{\partial y}\right), \quad \left(\frac{\partial y}{\partial \phi}\right) = -J\left(\frac{\partial \psi}{\partial x}\right), \quad \left(\frac{\partial x}{\partial \psi}\right) = -J\left(\frac{\partial \phi}{\partial y}\right), \quad \left(\frac{\partial y}{\partial \psi}\right) = J\left(\frac{\partial \phi}{\partial x}\right) \qquad (22.04)$$

provided

$$0 < |J| < \infty$$

where J denotes the Jacobian

$$J = \left(\frac{\partial x}{\partial \phi}\right)\left(\frac{\partial y}{\partial \psi}\right) - \left(\frac{\partial x}{\partial \psi}\right)\left(\frac{\partial y}{\partial \phi}\right) \qquad (22.05)$$

Equations (22.03) and (22.05) give us

$$J = \pm W \text{ where } W = \sqrt{EG - F^2} \qquad (22.06)$$

Let β be the angle made by the tangent to the co-ordinate line ϕ = constant, directed in the sense of increasing ψ, with x-axis. From the third equation of (22.03), we write

$$\frac{\partial x}{\partial \psi} = \sqrt{G}\cos\beta \;,\; \frac{\partial y}{\partial \psi} = \sqrt{G}\sin\beta \tag{22.07}$$

Substitution of $\frac{\partial x}{\partial \psi}$ and $\frac{\partial y}{\partial \psi}$ from (22.07) in the second equation of (22.03) yields

$$F = \sqrt{G}\left[\frac{\partial x}{\partial \phi}\cos\beta + \frac{\partial y}{\partial \phi}\sin\beta\right] \tag{22.08}$$

Eliminating $\frac{\partial y}{\partial \phi}$ between (22.08) and the first equation of (22.03) and solving for $\frac{\partial x}{\partial \phi}$, we obtain

$$\frac{\partial x}{\partial \phi} = \frac{F}{\sqrt{G}}\cos\beta + \frac{J}{\sqrt{G}}\sin\beta \tag{22.09}$$

First equation of (22.03) and (22.09) require

$$\frac{\partial y}{\partial \phi} = \frac{F}{\sqrt{G}}\sin\beta - \frac{J}{\sqrt{G}}\cos\beta \tag{22.10}$$

From (22.07), (22.09), (22.10) and the conditions that the second order mixed derivatives of x and y with respect to ϕ and ψ are independent of the order of differentiation, we find that

$$\frac{\partial \beta}{\partial \phi} = \frac{J}{G}\gamma^2_{12} \;,\; \frac{\partial \beta}{\partial \psi} = \frac{J}{G}\gamma^2_{11} \tag{22.11}$$

where

$$\left.\begin{aligned} \gamma^2_{11} &= \frac{1}{2W^2}\left[F\frac{\partial G}{\partial \psi} - 2G\frac{\partial F}{\partial \psi} + G\frac{\partial G}{\partial \phi}\right] \\ \gamma^2_{12} &= \frac{1}{2W^2}\left[F\frac{\partial G}{\partial \phi} - G\frac{\partial E}{\partial \psi}\right] \end{aligned}\right\} \tag{22.12}$$

Similarly, if we let α denote the angle between the tangent to the co-ordinate line ψ = constant, directed in the sense of increasing ϕ, with x-axis, we find that

$$\left.\begin{aligned} \frac{\partial x}{\partial \phi} &= \sqrt{E}\cos\alpha \quad , \quad \frac{\partial y}{\partial \phi} = \sqrt{E}\sin\alpha \\ \frac{\partial x}{\partial \psi} &= \frac{F}{\sqrt{E}}\cos\alpha - \frac{J}{\sqrt{E}}\sin\alpha \\ \frac{\partial y}{\partial \psi} &= \frac{F}{\sqrt{E}}\sin\alpha + \frac{J}{\sqrt{E}}\cos\alpha \end{aligned}\right\} \qquad (22.13)$$

From (22.13) and the same conditions that the second order mixed derivatives of x and y with respect to ϕ and ψ are independent of the order of differentiation, we get

$$\frac{\partial \alpha}{\partial \phi} = \frac{J}{E}\Gamma^2_{11} \quad , \quad \frac{\partial \alpha}{\partial \psi} = \frac{J}{E}\Gamma^2_{12} \qquad (22.14)$$

where

$$\left.\begin{aligned} \Gamma^2_{11} &= \frac{1}{2W^2}\left[-F\frac{\partial E}{\partial \phi} + 2E\frac{\partial F}{\partial \phi} - E\frac{\partial E}{\partial \psi}\right] \\ \Gamma^2_{12} &= \frac{1}{2W^2}\left[E\frac{\partial G}{\partial \phi} - F\frac{\partial E}{\partial \psi}\right] \end{aligned}\right\} \qquad (22.15)$$

Equations (22.11) together with the condition

$$\frac{\partial^2 \beta}{\partial \phi \partial \psi} = \frac{\partial^2 \beta}{\partial \psi \partial \phi}$$

implies

$$\frac{\partial}{\partial \psi}\left[\frac{J}{G}\gamma^2_{12}\right] - \frac{\partial}{\partial \phi}\left[\frac{J}{G}\gamma^2_{11}\right] = 0 \qquad (22.16)$$

We now show that if (22.16) is satisfied, then the functions $x(\phi,\psi)$ and $y(\phi,\psi)$ can be obtained in terms of E, F and G where E, F, G satisfy (22.02).

Equation (22.16) implies the existence of a function $\beta(\phi\ \psi)$ such that

$$\frac{\partial \beta}{\partial \phi} = \frac{J}{G}\gamma^2_{12} \quad , \quad \frac{\partial \beta}{\partial \psi} = \frac{J}{G}\gamma^2_{11} \qquad (22.17)$$

Therefore, we have

$$\beta = \int \frac{J}{G} \{ \gamma^2_{12} \, d\phi + \gamma^2_{11} d\psi \} \tag{22.18}$$

The functions $x(\phi,\psi)$ and $y(\phi,\psi)$ are given by

$$x = \int \{ (\frac{\partial x}{\partial \phi}) \, d\phi + (\frac{\partial x}{\partial \psi}) \, d\psi \} \text{ and } y = \int \{ (\frac{\partial y}{\partial \phi}) \, d\phi + (\frac{\partial y}{\partial \psi}) \, d\psi \} \tag{22.19}$$

Making use of complex variable $z = x + iy$, we find

$$z = \int \frac{e^{i\beta}}{\sqrt{G}} \{ (F - iJ) \, d\phi + G \, d\psi \} \tag{22.20}$$

where β is given by (22.18).

If E, F, G are given functions of ϕ and ψ, then the system

$$x = x(\phi,\psi) \quad , \quad y = y(\phi,\psi)$$

will serve as curvilinear co-ordinate system if and only if

$$\frac{\partial}{\partial \psi} \left[\frac{J}{G} \gamma^2_{12} \right] - \frac{\partial}{\partial \phi} \left[\frac{J}{G} \gamma^2_{11} \right] = 0$$

When the above condition is satisfied, the functions $x(\phi,\psi)$, $y(\phi,\psi)$ can be obtained from (22.20) where β is given by (22.18).

From the relation

$$W = \sqrt{EG - F^2}$$

we find that

$$\left.\begin{aligned} \frac{\partial}{\partial \phi} \left(\frac{G}{2W^2} \right) &= \frac{1}{W^2} (G \gamma^2_{22} - F \gamma^2_{12}) \\ \frac{\partial}{\partial \psi} \left(\frac{G}{2W^2} \right) &= \frac{1}{W^2} (G \gamma^2_{12} - F \gamma^2_{11}) \end{aligned}\right\} \tag{22.21}$$

and

$$\frac{\partial}{\partial\phi}\left(\frac{G}{W}\right) - \frac{\partial}{\partial\psi}\left(\frac{F}{W}\right) = \frac{1}{W}(G\ \gamma^2_{22} - 2F\ \gamma^2_{12} + E\ \gamma^2_{11}) \tag{22.22}$$

where γ^2_{11}, γ^2_{12} are given by (22.12) and

$$\gamma^2_{22} = \frac{1}{2W^2}\left\{ -G\frac{\partial E}{\partial\phi} + 2F\frac{\partial F}{\partial\phi} - F\frac{\partial E}{\partial\psi}\right\} \tag{22.23}$$

Similarly, we have

$$\left.\begin{aligned} \frac{\partial}{\partial\phi}\left(\frac{E}{2W^2}\right) &= \frac{1}{W^2}(F\ \Gamma^2_{11} - E\ \Gamma^2_{12}) \\ \frac{\partial}{\partial\psi}\left(\frac{E}{2W^2}\right) &= \frac{1}{W^2}(F\ \Gamma^2_{12} - E\ \Gamma^2_{22}) \end{aligned}\right\} \tag{22.24}$$

and

$$\frac{\partial}{\partial\phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial\psi}\left(\frac{E}{W}\right) = \frac{1}{W}[G\ \Gamma^2_{11} - 2F\ \Gamma^2_{12} + E\ \Gamma^2_{22}] \tag{22.25}$$

where Γ^2_{11} , Γ^2_{12} are given by (22.15) and

$$\Gamma^2_{22} = \frac{1}{2W^2}\left\{E\frac{\partial G}{\partial\psi} - 2F\frac{\partial F}{\partial\psi} + F\frac{\partial G}{\partial\phi}\right\} \tag{22.26}$$

CHAPTER III

ON ALIGNED FLOWS

Section 1. Flow Equations

In this chapter, we consider flows in which magnetic field vector and the velocity field vector are everywhere parallel. We assume the fluid under consideration to be steady, inviscid, compressible, thermally non-conducting and electrically infinitely conducting. The governing equations for these flows are:

$$\operatorname{div} (\rho \vec{V}) = 0 \tag{31.01}$$

$$\rho (\vec{V} \cdot \operatorname{grad}) \vec{V} + \operatorname{grad} p = \mu (\operatorname{curl} \vec{H}) \times \vec{H} \tag{31.02}$$

$$\vec{V} \cdot \operatorname{grad} s = 0 \tag{31.03}$$

$$\rho = \rho (p,s) \tag{31.04}$$

$$\operatorname{div} \vec{H} = 0 \tag{31.05}$$

$$\vec{H} = \nu \vec{V} \tag{31.06}$$

where ν is a scalar function.

From (31.01), (31.05) and (31.06), we get

$$\vec{H} = \alpha \rho \vec{V} \tag{31.07}$$

where α is a scalar function such that

$$\vec{V} \cdot \operatorname{grad} \alpha = 0 \tag{31.08}$$

and

$$\alpha = \frac{\nu}{\rho} \tag{31.09}$$

Section 2(a). Substitution Principle.

The substitution principle for inviscid, thermally and electrically non-conducting fluids having an equation of state of the form $\rho = P(p)\ S(s)$ was established by R.C. Prim (1952). O.P. Chandna (1971) found that the substitution principle holds for non-conducting fluids having arbitrary equation of state provided ρ is constant on an individual streamline. P. Smith (1963) showed that the substitution principle could be extended to MFD flows of inviscid, thermally non-conducting fluids having a product form of equation of state. In this section, we find the condition that the substitution principle holds for electrically infinitely conducting fluids with arbitrary equation of state for aligned flows.

Equations (31.01) to (31.06) are a set of ten scalar equations in ten dependent scalar variables, the three components of the velocity vector $\vec{V}$, the three components of the magnetic field vector $\vec{H}$, the three thermodynamic variables p, ρ and s, and the scalar function ν. The equivalent problem to this system of equations is determination of seven unknowns $\vec{V}$, ρ, p, s and α from seven equations (31.01) to (31.04) and (31.08). By use of state equation (31.04) one of the thermodynamic variables can be eliminated, giving six equations in six dependent variables. In the following work, we show that for an arbitrary equation of state, these seven equations can be reduced to five equations in five dependent variables provided the density is constant along each individual streamline or the equation of state is of the product form.

Introducing a new variable $\vec{q} = \sqrt{\rho}\,\vec{V}$, we find that $\vec{H} = \lambda\,\vec{q}$ with $\lambda = \dfrac{\nu}{\sqrt{\rho}} = \alpha\sqrt{\rho}$ and the equations (31.01), (31.02), (31.05) reduce to:

$$\operatorname{div} \vec{q} + \frac{1}{2} (\vec{q} \cdot \operatorname{grad} \ln \rho) = 0 \tag{32.01}$$

$$(\vec{q} \cdot \operatorname{grad}) \vec{q} - \frac{1}{2} (\vec{q} \cdot \operatorname{grad} \ln \rho)\vec{q} + \operatorname{grad} p = \mu \lambda (\operatorname{curl} \lambda \vec{q}) \times \vec{q} \tag{32.02}$$

$$\operatorname{div} (\lambda \vec{q}) = 0 \tag{32.03}$$

From (31.03) and (31.04), we have

$$\vec{q} \cdot \operatorname{grad} \ln \rho = \left[\frac{\partial}{\partial p} (\ln \rho)\right]_s \vec{q} \cdot \operatorname{grad} p \tag{32.04}$$

Now, if either $\left[\frac{\partial}{\partial p} (\ln \rho)\right]_s$ is a function of p alone or $\vec{q} \cdot \operatorname{grad} \ln \rho = 0$, the equation (32.04) implies that (32.01) to (32.03) are five equations in five unknowns. The first possibility leads to the conclusion that the equation of state of the fluid is of the form $\rho = P(p)\,S(s)$. In the second case, $\vec{q} \cdot \operatorname{grad} \rho = 0$, and the five equations in five unknowns are

$$\operatorname{div} \vec{q} = 0 \tag{32.05}$$

$$(\vec{q} \cdot \operatorname{grad}) \vec{q} + \operatorname{grad} p = \mu \lambda (\operatorname{curl} \lambda \vec{q}) \times \vec{q} \tag{32.06}$$

$$\vec{q} \cdot \operatorname{grad} \lambda = 0 \tag{32.07}$$

The equations will be unchanged by change of dependent variables leaving $\vec{q}$, p and λ unchanged. This establishes:

<u>The Substitution Principle.</u> For conducting compressible fluid

having an arbitrary equation of state, any aligned flow field satisfying the flow equations is a member of an infinite family of flow fields sharing the same streamlines, the same pressure field, the same magnetic field provided the density is constant along each streamline and the members of the group are related by

$$p^* = p; \ \rho^* = m^2 \ \rho; \ \vec{V}^* = \frac{\vec{V}}{m}; \ \nu^* = m\nu; \ \rho(p^*,s^*) = m^2 \ \rho(p,s)$$

where m is any scalar function constant along each individual streamline.

<u>Section 2(b)</u>. <u>Geometric Implications of Substitution Principle for Plane Flows with Arbitrary Equation of State</u>.

By assumption

$$\vec{V} \ . \ \text{grad}\ \rho = 0 \tag{32.08}$$

Equations (31.03), (31.04) and (32.08) require

$$\vec{V} \ . \ \text{grad}\ p = 0 \tag{32.09}$$

The multiplication of equation (31.02) scalarly by $\vec{V}$ and use of equations (31.07) and (32.09) gives

$$\vec{V} \ . \ \text{grad}\ V = 0$$

or

$$v_1^2 \ \frac{\partial v_1}{\partial x} + v_1 \ v_2 \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) + v_2^2 \ \frac{\partial v_2}{\partial y} = 0 \tag{32.10}$$

where $\vec{V} = (v_1, v_2)$ and $|\vec{V}| = V$.

Using (32.08) in (31.01), we get

$$\text{div}\ \vec{V} = 0$$

or

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{32.11}$$

The equation (32.11) is satisfied by putting

$$v_1 = \frac{\partial\psi}{\partial y}, \quad v_2 = -\frac{\partial\psi}{\partial x} \tag{32.12}$$

Equation (32.10), using (32.12) can be written as

$$p\,q\,r + (q^2 - p^2)s - p\,q\,t = 0 \tag{32.13}$$

where

$$p = \left(\frac{\partial\psi}{\partial x}\right)_y; \quad q = \left(\frac{\partial\psi}{\partial y}\right)_x; \quad r = \left(\frac{\partial^2\psi}{\partial x^2}\right)_y$$

$$s = \left(\frac{\partial}{\partial x}\left[\left(\frac{\partial\psi}{\partial y}\right)_x\right]\right)_y; \quad t = \left(\frac{\partial^2\psi}{\partial y^2}\right)_x \tag{32.14}$$

Integration of (32.13) by Monge's Method yields

$$p^2 + q^2 = f(\psi) \tag{32.15}$$

where $f(\psi)$ is an arbitrary function of ψ.

On solving (32.15) by the method of characteristics, we find that the complete integral is given by

$$F(\psi) = x\cos\theta + y\sin\theta + \beta \tag{32.16}$$

where θ and β are two parameters. Equation (32.16) suggests that the streamlines are a family of parallel straight lines. The general integral is obtained by eliminating θ between

$$\left.\begin{aligned} F(\psi) &= x\cos\theta + y\sin\theta + \phi(\theta) \\ &\text{and} \\ 0 &= -x\sin\theta + y\cos\theta + \phi'(\theta) \end{aligned}\right\} \tag{32.17}$$

Eliminating $\vec{H}$ between (31.02) and (31.07), and using (31.08) and (32.08), we find

$$\rho\,(1 - \mu\,\alpha^2\,\rho)\,(\vec{V}\cdot\text{grad})\,\vec{V} + \text{grad}\left(p + \mu\,\frac{\alpha^2\rho^2V^2}{2}\right) = \vec{0} \tag{32.18}$$

Taking curl of (32.18) and using the fact that α, ρ, p, V are constant on each streamline, we get

$$\text{curl}\ [(\vec{V}\ .\ \text{grad})\ \vec{V}] = 0$$

or

$$\frac{\partial}{\partial x}\left(v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y}\right) = \frac{\partial}{\partial y}\left(v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y}\right) \tag{32.19}$$

if $\mu\ \alpha^2\ \rho \neq 1$.

Differentiating the 1st equation of (32.17) with respect to x and y respectively and using the second equation of (32.17), we obtain

$$p\ F'(\psi) = \cos\theta \quad , \quad q\ F'(\psi) = \sin\theta \tag{32.20}$$

Differentiations of equations (32.20) give:

$$F'(\psi)\ r + p^2\ F''(\psi) = -\sin\theta\ \frac{\partial\theta}{\partial x} \tag{32.21}$$

$$F'(\psi)\ t + q^2\ F''(\psi) = \cos\theta\ \frac{\partial\theta}{\partial y} \tag{32.22}$$

$$F'(\psi) s + F''(\psi)\ p\ q = -\sin\theta\ \frac{\partial\theta}{\partial y} = \cos\theta\ \frac{\partial\theta}{\partial x} \tag{32.23}$$

Further differentiations of these equations give:

$$F'(\psi)\ \frac{\partial r}{\partial x} + 3F''(\psi)\ p\ r + F'''(\psi)\ p^3 = -\sin\theta\ \frac{\partial^2\theta}{\partial x^2} - \cos\theta\ \left(\frac{\partial\theta}{\partial x}\right)^2 \tag{32.24}$$

$$F'(\psi)\ \frac{\partial t}{\partial y} + 3F''(\psi)\ q\ t + F'''(\psi)\ q^3 = \cos\theta\ \frac{\partial^2\theta}{\partial y^2} - \sin\theta\ \left(\frac{\partial\theta}{\partial y}\right)^2 \tag{32.25}$$

$$F'(\psi)\ \frac{\partial s}{\partial x} + F''(\psi)\ [2p\ s + r\ q] + p^2\ q\ F'''(\psi) = \cos\theta\ \frac{\partial^2\theta}{\partial x^2} - \sin\theta\ \left(\frac{\partial\theta}{\partial x}\right)^2 \tag{32.26}$$

$$F'(\psi)\ \frac{\partial s}{\partial y} + F''(\psi)\ [2q\ s + p\ t] + p\ q^2\ F'''(\psi) = -\sin\theta\ \frac{\partial^2\theta}{\partial y^2} - \cos\theta\ \left(\frac{\partial\theta}{\partial y}\right)^2 \tag{32.27}$$

Multiplying (32.24) by $\sin\theta$, (32.26) by $\cos\theta$ and subtracting, we have

$$\frac{\partial^2\theta}{\partial x^2} = [F\ (\psi)]^2 \left\{ p\ \frac{\partial s}{\partial x} - q\ \frac{\partial r}{\partial x} \right\} + 2F''(\psi)\ F'(\psi)\ (p^2 s - p\ q\ r) \tag{32.28}$$

Similarly, multiplying (32.25) by $\cos\theta$, (32.27) by $\sin\theta$ and subtracting, we find that

$$\frac{\partial^2\theta}{\partial y^2} = [F'(\psi)]^2 \left\{ p\ \frac{\partial t}{\partial y} - q\ \frac{\partial s}{\partial y} \right\} + 2F''(\psi)F'(\psi)(p\ q\ t - q^2 s) \tag{32.29}$$

Equation (32.28) together with (32.29) and (32.13) yields

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = [F'(\psi)]^2 \left\{ p\left(\frac{\partial s}{\partial x} + \frac{\partial t}{\partial y}\right) - q\left(\frac{\partial r}{\partial x} + \frac{\partial s}{\partial y} \right) \right\} \tag{32.30}$$

Using (32.19) in (32.30), we get

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{32.31}$$

Differentiating the second equation of (32.17) twice with respect to x and twice with respect to y respectively, we obtain

$$- \cos\theta\ \frac{\partial\theta}{\partial x} - \{ x\cos\theta + y\sin\theta - \phi''(\theta)\} \frac{\partial^2\theta}{\partial x^2}$$
$$+ \{x\sin\theta - y\cos\theta + \phi'''(\theta)\} \left(\frac{\partial\theta}{\partial x} \right)^2 = 0 \tag{32.32}$$

and

$$- \sin\theta\ \frac{\partial\theta}{\partial y} - \{ x\cos\theta + y\sin\theta - \phi''(\theta)\} \frac{\partial^2\theta}{\partial y^2}$$
$$+ \{x\sin\theta - y\cos\theta + \phi'''(\theta)\} \left(\frac{\partial\theta}{\partial y} \right)^2 = 0 \tag{32.33}$$

Adding (32.32) and (32.33), using (32.17), (32.23) and (32.31), we find

$$[\phi'''(\theta) + \phi'(\theta)] \left\{ \left(\frac{\partial\theta}{\partial x} \right)^2 + \left(\frac{\partial\theta}{\partial y} \right)^2 \right\} = 0 \tag{32.34}$$

This equation (32.34) implies that either $(\frac{\partial\theta}{\partial x})^2 + (\frac{\partial\theta}{\partial y})^2 = 0$ or $\phi'''(\theta) + \phi'(\theta) = 0$. The first possibility gives θ = constant i.e. the streamlines are a family of parallel straight lines. From the second possibility, we have

$$\phi(\theta) = A \sin\theta + B \cos\theta + C \tag{32.35}$$

where A, B and C are arbitrary constants.
Substituting for $\phi(\theta)$ in (32.17), we have

$$F(\psi) = (x + B) \cos\theta + (y + A) \sin\theta + C$$

and

$$0 = -(x + B) \sin\theta + (y + A) \cos\theta$$

Squaring and adding, we get

$$[F(\psi) - C]^2 = (x + B)^2 + (y + A)^2$$

which implies that the streamlines are concentric circles.
Hence, we have

Theorem 3.1 If the substitution principle holds for plane flows with arbitrary equation of state, then the only possible flow fields are the general vortex flows or flows in parallel straight lines.

Section 3. Some Properties of Plane Flows.

(A) Sonic Flows. By assumption

$$V = C(\rho,s) \tag{33.01}$$

where $C^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$. This quantity C is called the speed of sound. Scalar multiplication of (31.02) by $\vec{V}$ yields

$$\rho \vec{V} \cdot \text{grad } \frac{V^2}{2} + \vec{V} \cdot \text{grad } p = 0 \tag{33.02}$$

or

$$\rho V \frac{\partial V}{\partial \rho} \vec{V} \cdot \text{grad } \rho + C^2 \vec{V} \cdot \text{grad } \rho = 0$$

Using (33.01), we get

$$\left\{\rho C \frac{\partial C}{\partial \rho} + C^2\right\} \vec{V} \cdot \text{grad } \rho = 0$$

Since $(\rho \frac{\partial C}{\partial \rho} + C)$ is strictly positive when $\left(\frac{\partial^2 p}{\partial \rho^2}\right)_s > 0$, therefore we have

$$\vec{V} \cdot \text{grad } \rho = 0$$

Using the same proof as in Section 2(b) of Chapter III, we conclude

Theorem 3.2. If the speed is sonic in a plane flow, then the streamlines are concentric circles or a family of parallel straight lines.

(B) Vorticity.

In section 2 of this chapter we have shown that $\vec{V} \cdot \text{grad } \rho = 0$ implies $\vec{V} \cdot \text{grad } p = 0$ and $\vec{V} \cdot \text{grad } V = 0$. Let

$$\vec{V} \cdot \text{grad } p = 0$$

which together with the equations (31.03) and (31.04) requires that $\vec{V} \cdot \text{grad } \rho = 0$.

When $\vec{V} \cdot \text{grad } V = 0$, the equation (33.02) reduces to $\vec{V} \cdot \text{grad } p = 0$.

Therefore, if one of the three scalars V, p, ρ is constant along each streamline or if the speed is everywhere sonic, then the other two are constant along each streamline.

Since $C = C(\rho, s)$, using (31.03), we have

$$\vec{V} \cdot \text{grad } C = 0 \iff \vec{V} \cdot \text{grad } \rho = 0$$

The specific enthalpy i is a function of p, s. Therefore i is a constant on each individual streamline provided one of the three variables V, p or C is constant on each streamline or V = C everywhere.

Using (31.07) in (31.02) and simplifying the equation thus obtained, we get

$$\frac{1}{\rho} \text{ grad } p = (1 - \mu\, \alpha^2 \rho)\ (\vec{\omega} \times \vec{V} + V \text{ grad } V) + \mu\, \alpha\, V \text{ grad } (\alpha\, \rho\, V) \tag{33.03}$$

where the vorticity vector $\vec{\omega}$ is given by

$$\vec{\omega} = \text{curl } \vec{V} .$$

Eliminating $(\frac{1}{\rho}$ grad p) between (33.03) and the relation

$$\frac{1}{\rho} \text{ grad } p = \text{grad } i - T \text{ grad } s \tag{33.04}$$

we find that

$$(1 - \mu\, \alpha^2 \rho)\ (\vec{\omega} \times \vec{V} + V \text{ grad } V) + \mu\, \alpha\, V \text{ grad } (\alpha\, \rho\, V) = \text{grad } i - T \text{ grad } s \tag{33.05}$$

Assuming that V and ρ are constants on each streamline, (33.05) gives

$$(1 - \mu\, \alpha^2\, \rho) \left\{ \omega\, V + V\, \frac{\partial V}{\partial n} \right\} + \mu\, \alpha\, V \frac{\partial}{\partial n}\, (\alpha\, \rho\, V) = \frac{\partial i}{\partial n} - T\, \frac{\partial s}{\partial n} \tag{33.06}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the normal to the streamline and $\omega = |\vec{\omega}|$.

Differentiating (33.06) along the streamline, we get

$$\frac{\partial \omega}{\partial s} = 0$$

Hence the result:

<u>Theorem 3.3.</u> If one of the four scalar quantities V, p, ρ or C is constant along each streamline, or if the speed is everywhere sonic, then the other three together with the vorticity ω, are also constant along each streamline.

(C) <u>Circular or Straight Streamlines</u>.

To investigate the general vortex flows we take polar co-ordinate system as the streamline co-ordinate system. Equations (31.01) and (31.02) can be written as

$$\frac{\partial}{\partial \theta} (\rho\ V) = 0 \qquad (33.07)$$

$$\rho\ V \frac{\partial V}{\partial \theta} + \frac{\partial p}{\partial \theta} = 0 \qquad (33.08)$$

$$\frac{\rho\ V^2}{r} - \frac{\partial p}{\partial r} = \mu \frac{\alpha \rho V}{r} \frac{\partial}{\partial r} (r\ \alpha\ \rho\ V) \qquad (33.09)$$

where r = constant are the streamlines and θ = constant are their orthogonal trajectories.

From (33.07) and (33.08), we obtain

$$\frac{\partial V}{\partial \theta} (V^2 - C^2) = 0 \qquad (33.10)$$

This implies that either velocity magnitude is constant along each streamline or the flow is sonic. Likewise, if the streamlines are straight and parallel lines, then we take the rectangular co-ordinate system and obtain the same result. Hence, we have

<u>Theorem 3.4</u>. If the flow is general vortex or parallel flow, then either the flow is sonic or the velocity magnitude is constant along each streamline.

Section 4(A). On the Uniqueness of Incompressible Flows.

In this section, we find the physical condition for two dynamically distinct flows to have the same streamline pattern and study the geometric implications. For non-conducting fluids, study of such flows has been done by D. Gilbarg (1947), R.C. Prim (1949), J.L. Ericksen (1952).

For incompressible flows, equation (31.01) reduces to

$$\operatorname{div} \vec{V} = 0 \tag{34.01}$$

From (31.02) and (31.07), it is seen that

$$\rho(1 - \mu\, \alpha^2\rho)\ (\vec{V}\ .\ \operatorname{grad})\ \vec{V} + \operatorname{grad}\left(p + \mu\,\frac{\alpha^2\rho^2V^2}{2}\right) = \vec{0} \tag{34.02}$$

Taking curl of both sides of (34.02), we get

$$\rho(1 - \mu\, \alpha^2\rho)\ \operatorname{curl}[(\vec{V}.\ \operatorname{grad})\vec{V}] - 2\mu\ \rho^2\alpha\ \operatorname{grad}\ \alpha \times [(\vec{V}.\ \operatorname{grad})\vec{V}] = \vec{0} \tag{34.03}$$

Equations (34.01), (31.08) and (34.03) are the necessary and sufficient conditions satisfied by $\vec{V}$ and α. We investigate the problem: 'Given a velocity field $\vec{V}$ satisfying (34.01) and (34.03) under what conditions will the velocity field $\vec{V}* = f\ \vec{V}$ satisfy the same equations?'

From equation (34.01), we find

$$\vec{V}\ .\ \operatorname{grad}\ f = 0 \tag{34.04}$$

Equation (34.03) gives

$$\rho(1 - \mu\, \alpha^2\rho)\ \operatorname{curl}\ \{f\ (\vec{V}\ .\ \operatorname{grad})\ (f\ \vec{V})\}$$
$$- 2\mu\ \rho^2\alpha\ \operatorname{grad}\ \alpha \times \{f\ (\vec{V}\ .\ \operatorname{grad})\ (f\ \vec{V})\} = \vec{0}$$

i.e.

$$\rho(1 - \mu\, \alpha^2\rho)\ \operatorname{curl}\ \{f^2(\vec{V}\ .\ \operatorname{grad})\vec{V} + \vec{V}\ f\ (\vec{V}\ .\ \operatorname{grad}\ f)\}$$
$$- 2\mu\ \rho^2\alpha\ \operatorname{grad}\ \alpha \times \{f^2(\vec{V}\ .\ \operatorname{grad})\vec{V} + \vec{V}\ f\ (\vec{V}\ .\ \operatorname{grad}\ f)\} = \vec{0}$$

Using (34.04), we obtain

$$\rho(1 - \mu\,\alpha^2\rho)\ \text{curl}\ \{f^2(\vec{V}\ .\ \text{grad})\vec{V}\}$$
$$-\ 2\mu\,\rho^2\alpha\ \text{grad}\ \alpha \times \{f^2(\vec{V}\ .\ \text{grad})\vec{V}\} = \vec{0}$$

or

$$\rho(1 - \mu\,\alpha^2\rho)\ \text{grad}\ f^2 \times (\vec{V}\ .\ \text{grad})\vec{V} + f^2[\rho(1 - \mu\,\alpha^2\rho)\ \text{curl}\{(\vec{V}\ .\ \text{grad})\vec{V}\}$$
$$-\ 2\mu\,\rho^2\alpha\ \text{grad}\ \alpha \times (\vec{V}\ .\ \text{grad})\vec{V}] = \vec{0}$$

Using (34.03) in this equation, we find that

$$\rho(1 - \mu\,\alpha^2\rho)\ \text{grad}\ f^2 \times (\vec{V}\ .\ \text{grad})\vec{V} = \vec{0} \qquad (34.05)$$

From (34.02) it is seen that (34.05) implies

$$\text{grad}\ f \times \text{grad}\left(p + \mu\,\frac{\alpha^2\rho^2V^2}{2}\right) = \vec{0} \qquad (34.06)$$

Scalar multiplication of (34.02) by $\vec{V}$, gives

$$\vec{V}\ .\ \text{grad}\left(\rho\,\frac{V^2}{2} + p\right) = 0$$

i.e.

$$\rho\,\frac{V^2}{2} + p = m \qquad (34.07)$$

where

$$\vec{V}\ .\ \text{grad}\ m = 0$$

Eliminating p between (34.06) and (34.07), we get

$$\text{grad}\ f \times \text{grad}\left(m - \frac{\rho V^2}{2} + \mu\,\frac{\alpha^2\rho^2V^2}{2}\right) = \vec{0} \qquad (34.08)$$

Since the functions f, m and α are constant along each individual streamline, we get from (34.08), that

$$(1 - \mu\,\rho\,\alpha^2)\ \text{grad}\ f \times \text{grad}\ V = \vec{0}$$

For flow regions where $1 - \mu\,\rho\,\alpha^2 \neq 0$, we find that $\vec{V}^* = f\,(x,y,z)\vec{V}$ satisfies (34.01) and (34.03) provided

$$\text{grad}\ f \times \text{grad}\ V = \vec{0} \qquad (34.09)$$

The condition (34.09) allows one of the following three possibilities:

(a) grad f = $\vec{0}$. In this case $\vec{V}^*$ is proportional to V so that f has any assigned constant value.

(b) grad f $\neq \vec{0}$, grad V = $\vec{0}$. In this case the flow is with uniform constant velocity magnitude and with uniform pressure and hence a flow with straight streamlines.

(c) f = a(V) with grad f $\neq \vec{0}$, grad V $\neq \vec{0}$. In this case, (34.04) holds if and only if in the flow region the flow has a constant velocity magnitude along each individual streamline.

From the above study we can thus state the following result:

Theorem 3.5. Any flow in a region, where $1 - \mu\, \rho\, \alpha^2 \neq 0$, is unique unless it has a constant velocity magnitude along each individual streamline.

Section 4(B). Geometric Implications.

(i) For Plane Flows. If the speed is constant on each streamline, the velocity components v_1 and v_2 satisfy the equations (32.10), (32.11) and (32.19). Therefore the only possible flow fields are the general vortex flows or flows in parallel straight lines. This result together with Theorem 3.5 may be stated as

Theorem 3.6. Plane flows in a region, where $1 - \mu\, \alpha^2 \rho \neq 0$, are unique except those having concentric circles (or parallel straight lines) as streamlines.

(ii) For Axially-Symmetric Flows. We take r as the radius from the axis of symmetry and x as the co-ordinate along the axis.

Letting ψ as the stream function, the equation of continuity (34.01) gives

$$\frac{d\psi}{dn} = r\,V \tag{34.10}$$

where dn denotes the outward normal differential to the streamline. Taking $V = V(\psi)$, we get from (34.07) and (31.08), that $p = p(\psi)$ and $\alpha = \alpha(\psi)$. From the equation (34.02), we find that

$$\rho(1 - \mu\,\alpha^2\rho)V^2\kappa + \frac{d}{dn}\left\{p + \mu\,\frac{\alpha^2\rho^2V^2}{2}\right\} = 0 \tag{34.11}$$

where $\kappa(x,\psi)$ is the curvature of the streamline.

From (34.10) and (34.11), we have

$$\kappa(x,\psi) = -\frac{1}{\rho V^2(1 - \mu\,\alpha^2\rho)}\;\frac{d}{d\psi}\left\{p + \mu\,\frac{\alpha^2\rho^2V^2}{2}\right\}r\,V$$

i.e.

$$\kappa(x,\psi) = A(\psi)\;r(x,\psi) \tag{34.12}$$

where

$$A(\psi) = -\frac{1}{\rho\,V(1 - \mu\,\alpha^2\rho)}\;\frac{d}{d\psi}\left\{p + \mu\,\frac{\alpha^2\rho^2V^2}{2}\right\}$$

For convenience letting q denote the streamline slope,

$$q(x,\psi) \equiv \left(\frac{\partial r}{\partial x}\right)_\psi .$$

we have the familiar expression for κ

$$\kappa(x,\psi) = \frac{\left(\frac{\partial q}{\partial x}\right)_\psi}{(1 + q^2)^{3/2}} \tag{34.13}$$

From (34.12) and (34.13), we get

$$q\left(\frac{\partial q}{\partial r}\right)_\psi = A(\psi)\;(1 + q^2)^{3/2}\;r$$

which yields upon integration

$$(1 + q^2)^{1/2} = \frac{2}{A(\psi)r^2 + B(\psi)} \tag{34.14}$$

where $B(\psi)$ is an arbitrary function of ψ.

Equation (34.10) can be written as

$$d\psi = \frac{r\ V}{\sqrt{1+q^2}} \quad (dr - q\ dx)$$

or

$$dr = q\ dx + \frac{\sqrt{1+q^2}}{r\ V}\ d\psi \tag{34.15}$$

which suggests that the condition of integrability for the function $r(x,\psi)$ is

$$\left(\frac{\partial q}{\partial \psi}\right)_x = \frac{\partial}{\partial x}\left(\frac{\sqrt{1+q^2}}{r\ V}\right)_\psi$$

$$= \frac{q^2}{r\ V\sqrt{1+q^2}}\left(\frac{\partial q}{\partial r}\right)_\psi - \frac{\sqrt{1+q^2}}{V\ r^2}\ q \tag{34.16}$$

Also, by (34.15)

$$\left(\frac{\partial q}{\partial \psi}\right)_x = \left(\frac{\partial q}{\partial \psi}\right)_r + \frac{\sqrt{1+q^2}}{r\ V}\left(\frac{\partial q}{\partial r}\right)_\psi \tag{34.17}$$

From (34.16) and (34.17), we have

$$\left(\frac{\partial q}{\partial \psi}\right)_r + \frac{\sqrt{1+q^2}}{r\ V}\left(\frac{\partial q}{\partial r}\right)_\psi - \frac{q^2}{r\ V\sqrt{1+q^2}}\left(\frac{\partial q}{\partial r}\right)_\psi + \frac{\sqrt{1+q^2}}{V\ r^2}\ q = 0$$

Multiplying both the sides by q V, we get

$$\frac{V}{2}\left(\frac{\partial q^2}{\partial \psi}\right)_r + \frac{1}{2r\sqrt{1+q^2}}\left(\frac{\partial q^2}{\partial r}\right)_\psi + \frac{q^2\sqrt{1+q^2}}{r^2} = 0 \tag{34.18}$$

Substituting (34.14) into (34.18), we obtain the condition

$$r^4\left[2V(\psi)\ \frac{dA(\psi)}{d\psi} + 3A^2(\psi)\right] + 2r^2\left[V(\psi)\ \frac{dB(\psi)}{d\psi} + 2A(\psi)\ B(\psi)\right]$$

$$+ [B^2 - 4] = 0 \tag{34.19}$$

If $B^2 = 4$ and $A = 0$, the equation (34.19) is identically satisfied. Equation (34.14) requires that $q = 0$; i.e. that $r = r(\psi)$. In case $B^2 \neq 4$ or $A \neq 0$, the equation (34.19) can be solved for r in terms of the coefficients. But these coefficients are functions of ψ alone, so that again $r = r(\psi)$. This establishes:

Theorem 3.7. All axially symmetric flows in a region where $1 - \mu \rho \alpha^2 \neq 0$ are unique except purely axial flows.

(iii) For Spatial Irrotational Flows. For general spatial flows the geometric implications of Theorem 3.5 are unknown. However it has been proven by G. Hamel (1937) that if the flows are incompressible and irrotational in presence or absence of external force fields, then the only flows with constant velocity magnitude along each streamline are helicoidal flows.

Hence, we have:

Theorem 3.8. All irrotational flows in the flow region of Theorem 3.5 are unique except the helicoidal flows.

(iv) For Spatial Doubly-Laminar Flows. From the equation (34.07), we find, that constancy of velocity magnitude along each streamline implies the constancy of the pressure along an individual streamline. Taking the velocity magnitude, V, constant along each individual streamline, we find that the velocity field satisfies the equations

$$\text{div}\ \vec{V} = 0 \tag{34.20}$$

$$\vec{V} \cdot \text{grad}\ V = 0 \tag{34.21}$$

$$\rho(1 - \mu\ \alpha^2\rho)\ \text{curl}\ [(\text{curl}\ \vec{V})\times \vec{V}] - 2\mu\ \rho^2\alpha \times [(\text{curl}\ \vec{V})\times \vec{V}] - \mu\ \rho^2\alpha\ \text{grad}\ \alpha \times \text{grad}\ V^2 = \vec{0} \tag{34.22}$$

We let

$$\vec{V} = \lambda\ \text{grad}\ \phi \tag{34.23}$$

so that we study the doubly-laminar flow fields. The unit normal vector field for the family of surfaces $\phi(x,y,z) = \text{constant}$ are

$$\vec{n} = \frac{\text{grad}\ \phi}{|\text{grad}\ \phi|} \tag{34.24}$$

From (34.23) and (34.24) we write

$$\vec{V} = V\ \vec{n} \tag{34.25}$$

such that

$$V = \lambda\ |\text{grad}\ \phi|$$

and by (34.21)

$$|\text{grad}\ \phi|\ \vec{n}\cdot\text{grad}\ \lambda + \lambda\ \vec{n} \cdot \text{grad}\ (|\text{grad}\ \phi|) = 0 \tag{34.26}$$

Equations (34.20), (34.21) and (34.25) require

$$\text{div}\ \vec{n} = 0 \tag{34.27}$$

Hence, a necessary and sufficient condition for a velocity field satisfying (34.21), (34.24) and (34.25) to comply with (34.20) is that $\phi\ (x,y,z) = \text{constant}$ be minimal surfaces.

From (34.21) and (34.25), we find that

$$(\text{curl}\ \vec{V}) \times \vec{V} = V^2[(\text{curl}\ \vec{n}) \times \vec{n}] - \text{grad}\ \frac{V^2}{2} \tag{34.28}$$

Taking curl of both sides, we get

$$\text{curl } [(\text{curl } \vec{V})\times \vec{V}] = \text{grad } V^2\times[(\text{curl } \vec{n})\times \vec{n})$$

$$+ V^2 \text{ curl } [(\text{curl } \vec{n})\times \vec{n}] \qquad (34.29)$$

Using (34.24), we have

$$\text{curl } \vec{n} = \vec{n} \times \text{grad } (\ln|\text{grad } \phi|) \qquad (34.30)$$

so that

$$(\text{curl } \vec{n})\times \vec{n} = \text{grad } (\ln|\text{grad } \phi|) - \{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\}\vec{n}$$

Taking curl of both sides, we obtain

$$\text{curl } [(\text{curl } \vec{n})\times \vec{n}] = - \{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\} \text{ curl } \vec{n}$$

$$- \text{grad } \{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\}\times \vec{n}$$

Substituting for curl $\vec{n}$ from (34.30), we get

$$\text{curl } [(\text{curl } \vec{n})\times \vec{n}] = - \{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\}\{\vec{n} \times \text{grad } (\ln|\text{grad } \phi|)\} - \text{grad } \{\vec{n}\cdot \text{grad } (\ln|\text{grad } \phi|)\} \qquad (34.31)$$

Equation (34.21) requires that

$$\text{grad } V^2\times [(\text{curl } \vec{n})\times \vec{n}] = - [\text{curl } \vec{n} \cdot \text{grad } V^2]\vec{n}$$

Use of (34.30) gives

$$\text{grad } V^2\times [(\text{curl } \vec{n})\times \vec{n}] = - [\vec{n} \times \text{grad } (\ln|\text{grad } \phi|)\cdot \text{grad } V^2]\vec{n} \qquad (34.32)$$

Using (34.31) and (34.32) in (34.29), we find

$$\text{curl } [(\text{curl } \vec{V}) \times \vec{V}] = -V^2\{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\}\{\vec{n} \times \text{grad } (\ln|\text{grad } \phi|)$$
$$- V^2 \text{ grad } \{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\} \times \vec{n}$$
$$- [\vec{n} \times \text{grad } (\ln|\text{grad } \phi|) \cdot \text{grad } V^2]\vec{n} \tag{34.33}$$

Substituting (34.28) and (34.33) in (34.22), we get

$$2\mu\, \rho^2\alpha(\text{curl } \vec{n} \cdot \text{grad } \alpha)\vec{n} - \rho(1-\mu\, \alpha^2\rho)[V^2\{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\}$$
$$\{\vec{n} \times \text{grad } (\ln|\text{grad } \phi|)\} + V^2\text{grad}\{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\} \times \vec{n}$$
$$+ \{\vec{n} \times \text{grad } (\ln|\text{grad } \phi|) \cdot \text{grad } V^2\}\vec{n}] = \vec{0}$$

where (31.08) has been used. Elimination of curl $\vec{n}$ with the help of (34.30) gives

$$2\mu\, \rho^2\alpha\{\vec{n} \times \text{grad } (\ln|\text{grad } \phi|) \cdot \text{grad } \alpha\}\vec{n} - \rho(1-\mu\, \alpha^2\rho)$$
$$[V^2\{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\}\{\vec{n} \times \text{grad } (\ln|\text{grad } \phi|)\}$$
$$+ V^2 \text{ grad } \{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\} \times \vec{n}$$
$$+ \{\vec{n} \times \text{grad } (\ln|\text{grad } \phi|) \cdot \text{grad } V^2\}\vec{n}] = \vec{0}$$

Vector pre-multiplication with $\vec{n}$ gives

$$\{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\}\vec{n} \times \{\vec{n} \times \text{grad } (\ln|\text{grad } \phi|)\}$$
$$+ \vec{n} \times [\text{grad } \{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\} \times \vec{n}] = \vec{0}$$

i.e.

$$\vec{n} \times \{\vec{n} \times \text{grad } (\ln|\text{grad } \phi|)\} + \vec{n} \times [\text{grad } \{\ln (\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\} \times \vec{n}] = \vec{0}$$

or

$$\{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\}\vec{n} - \text{grad } (\ln|\text{grad } \phi|)$$
$$+ \text{grad } [\ln\{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\}] - [\vec{n} \cdot \text{grad } \{\ln (\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|))\}]\vec{n} = \vec{0}$$

or

$$\{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)\}\vec{n} + \text{grad}\left[\ln\left\{\frac{\vec{n} \cdot \text{grad } (\ln|\text{grad } \Phi|)}{|\text{grad } \phi|}\right\}\right]$$
$$- [\vec{n} \cdot \text{grad } \{\ln (\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|))\}]\vec{n} = \vec{0}$$

Another vector pre-multiplication with $\vec{n}$ yields

$$\vec{n} \times \text{grad} \left[\ln \left\{ \frac{\vec{n} \cdot \text{grad } (\ln|\text{grad } \phi|)}{|\text{grad } \phi|} \right\} \right] = \vec{0}$$

This condition implies [R.C. Prim (1949)] that the surfaces ϕ = constant can be so parametrised as to cut off equal arcs along each individual streamline.

Substitution of (34.24) in (34.27) shows that ϕ = constant are family of potential surfaces. The geometric implication of our analysis is the same as that of the study of R.C. Prim (1949) and G. Hamel (1937). The family of streamlines for doubly-laminar flow fields having a constant velocity magnitude along each streamline, is one parameter family of helical flow fields. We thus have the following theorem:

<u>Theorem 3.9.</u> All doubly-laminar spatial flows are unique except helical flows.

Section 5. On Plane Irrotational Compressible Flows.

The condition, that the flow is irrotational, is

$$\text{curl } \vec{v} = \vec{0} \tag{35.01}$$

For plane flows, equations (31.02) together with (31.07) can be written as

$$v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \alpha\, v_2 \frac{\partial}{\partial y} (\alpha\, \rho\, v_1) - \alpha\, v_2 \frac{\partial}{\partial x} (\alpha\, \rho\, v_2) \tag{35.02}$$

and

$$v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = \alpha\, v_1 \frac{\partial}{\partial x} (\alpha\, \rho\, v_2) - \alpha\, v_1 \frac{\partial}{\partial y} (\alpha\, \rho\, v_1) \tag{35.03}$$

Multiplying (35.02) with v_1, (35.03) with v_2 and adding, we get

$$v_1^2 \frac{\partial v_1}{\partial x} + v_1\, v_2 \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) + v_2^2 \frac{\partial v_2}{\partial y} + \frac{v_1}{\rho} \frac{\partial p}{\partial x} + \frac{v_2}{\rho} \frac{\partial p}{\partial y} = 0 \tag{35.04}$$

Eliminating ρ between (31.01) and (35.04), we obtain

$$(C^2 - v_1^2) \frac{\partial v_1}{\partial x} - v_1\, v_2 \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) + (C^2 - v_2^2) \frac{\partial v_2}{\partial y} = 0 \tag{35.05}$$

where $C^2 = (\frac{\partial p}{\partial \rho})_s$ is the square of the speed of sound.

Equations (35.01) and (35.05) are well known equations in plane gas dynamics. They are a set of hyperbolic partial differential equations for supersonic flows. If the flow is supersonic, the characteristic equations are given by

$$\left.\begin{aligned}\frac{\partial y}{\partial \alpha} &= \xi_+ \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial \beta} &= \xi_- \frac{\partial x}{\partial \beta} \\ \frac{\partial v_1}{\partial \alpha} &= - \xi_- \frac{\partial v_2}{\partial \alpha} \\ \frac{\partial v_1}{\partial \beta} &= - \xi_+ \frac{\partial v_2}{\partial \beta}\end{aligned}\right\} \tag{35.06}$$

where

$$\xi_\pm = \frac{- v_1 v_2 \pm C\sqrt{v_1^2 + v_2^2 - C^2}}{(C^2 - v_1^2)}$$

and α = constant, β = constant are the set of two characteristic curves.

Taking the fluid to be polytropic, we have

$$p = N \rho^\gamma \tag{35.07}$$

where N and γ are constant and the flow is assumed to be homentropic.

Equation (35.07) implies that

$$C^2 = \frac{\gamma p}{\rho} \tag{35.08}$$

Multiplying (31.02) scalarly by $\vec{V}$ and using (35.07), we find

$$V^2 + \frac{2\gamma}{\gamma-1} N^{1/\gamma} p^{\frac{\gamma-1}{\gamma}} = a^2 = \frac{2\gamma}{\gamma-1} N^{1/\gamma} p_o^{\frac{\gamma-1}{\gamma}} \tag{35.09}$$

where p_o is the stagnation pressure and a, p_o are constant on each streamline.

Equation (35.09) gives

$$p = \left(1 - \frac{V^2}{a^2}\right)^{\frac{\gamma}{\gamma-1}} p_o \tag{35.10}$$

From (35.07) and (35.10), we have

$$\rho = \left(1 - \frac{V^2}{a^2}\right)^{\frac{1}{\gamma-1}} \rho_o \tag{35.11}$$

Equations (35.07) and (35.09) yield

$$\frac{p_o}{\rho_o} = \frac{\gamma-1}{2\gamma} \; a^2 \tag{35.12}$$

Using (35.10), (35.11) and (35.12) in (35.08), we obtain

$$C^2 = \frac{(\gamma-1)\;(a^2 - v_1^2 - v_2^2)}{2} \tag{35.13}$$

These equations imply that we have reduced the problem of plane, compressible, homentropic, irrotational flows of MFD to that of plane, compressible, homentropic, irrotational flows of non-conducting gases. The set of equations, we have derived in this section, have been thoroughly studied in 'Supersonic Flow and Shock Waves' by R. Courant and K.O. Freidricks (1967).

Section 6. Plane Source Flows.

In this section, we find the general solution of plane source flows by the method used by O.P. Chandna (1968) to find the solutions of flow problems of non-conducting fluids. To solve the problem we take polar co-ordinate system as our natural co-ordinate system i.e. θ = constant are streamlines and r = constant their orthogonal trajectories. For this choice, flow equations take the form:

$$\frac{\partial}{\partial r}(r\ \rho\ V) = 0 \tag{36.01}$$

$$V\frac{\partial V}{\partial r} + \frac{1}{\rho}\ \frac{\partial p}{\partial r} = 0 \tag{36.02}$$

$$\frac{\partial p}{\partial \theta} + \mu\ \alpha\ \rho\ V\frac{\partial}{\partial \theta}(\alpha\ \rho\ V) = 0 \tag{36.03}$$

$$\frac{\partial s}{\partial r} = 0 \tag{36.04}$$

$$\frac{\partial \alpha}{\partial r} = 0 \tag{36.05}$$

Equation (36.01) gives

$$r\ \rho\ V = A(\theta) \tag{36.06}$$

where $A(\theta)$ is an arbitrary function of θ.

From (36.03) and (36.06), we get

$$\frac{\partial p}{\partial \theta} + \mu\frac{\alpha\ A(\theta)}{r}\ \frac{\partial}{\partial \theta}\left\{\alpha\ \frac{A(\theta)}{r}\right\} = 0$$

or

$$\frac{\partial p}{\partial \theta} + \frac{\mu}{r^2}\frac{\partial}{\partial \theta}\left\{\frac{\alpha^2 A^2(\theta)}{2}\right\} = 0 \tag{36.07}$$

Differentiating (36.07) partially with respect to r, we obtain

$$\frac{\partial^2 p}{\partial r \partial \theta} - \frac{2\mu}{r^3}\frac{\partial}{\partial \theta}\left\{\frac{\alpha^2 A^2(\theta)}{2}\right\} = 0 \tag{36.08}$$

because $\alpha = \alpha(\theta)$ only.

Equation (36.08), on using (36.07), becomes

$$\frac{\partial^2 p}{\partial r \partial \theta} + \frac{2}{r}\frac{\partial p}{\partial \theta} = 0 \tag{36.09}$$

Equation (36.09) is a linear hyperbolic partial differential equation of second order in canonical form wherein p is the dependent variable and ϕ,ψ are independent variables. Integrating (36.09) with respect to θ, we find

$$\frac{\partial p}{\partial r} + \frac{2p}{r} = B(r) \tag{36.10}$$

where B(r) is an arbitrary function of r.

Equation (36.10) can be rewritten as

$$r^2 \frac{\partial p}{\partial r} + 2r\,p = r^2\,B(r) \tag{36.11}$$

when $r \neq 0$.

Integration of (36.11) with respect to r yields

$$p\,r^2 = r^2\int B(r)\,dr + D(\theta)$$

or

$$p = E(r) + \frac{D(\theta)}{r^2} \tag{36.12}$$

where

$$E(r) = \frac{1}{r^2} \int r^2 B(r)\, dr$$

and $D(\theta)$ is an arbitrary function of θ.

Using (36.06) and (36.12), we get

$$\frac{A(\theta)}{r} \frac{\partial V}{\partial r} + E'(r) - \frac{2D(\theta)}{r^3} = 0$$

or

$$\frac{\partial V}{\partial r} = - \frac{1}{A(\theta)} \left[r\, E'(r) - \frac{2D(\theta)}{r^2} \right]$$

Integrating with respect to r, we obtain

$$V = - \frac{1}{A(\theta)} \left[F(r) + \frac{2D(\theta)}{r} \right] + M(\theta) \tag{36.13}$$

where

$$F(r) = \int r\, E'(r)\, dr$$

and $M(\theta)$ is an arbitrary function of θ.

From (36.06) and (36.13), we find

$$\rho \left[- \frac{1}{A(\theta)} \{ r\, F(r) + 2D(\theta) \} + r\, M(\theta) \right] = A(\theta)$$

or

$$\rho = \frac{A^2(\theta)}{rA(\theta)M(\theta) - \{ r\, F(r) + 2D(\theta) \}}$$

The solution thus obtained is not valid at the point $r = 0$.

This method can also be used to solve other flow problems.

CHAPTER IV

PLANE TRANSVERSE FLOWS

Section 1. Flow Equations.

Two dimensional MFD flow in the (x,y)-plane is said to be transverse flow if the vector field $\vec{H}$ is perpendicular to the plane of flow. In transverse flow the flow variables are independent of z.

The equations governing the steady motion of a fluid of infinite electrical conductivity moving under the influence of the solenoidal magnetic field vector $\vec{H}$ are

$$\operatorname{div} \rho \vec{V} = 0 \tag{41.01}$$

$$\rho(\vec{V} \cdot \operatorname{grad})\vec{V} + \operatorname{grad} p = \mu(\operatorname{curl} \vec{H}) \times \vec{H} \tag{41.02}$$

$$V. \operatorname{grad} s = 0 \tag{41.03}$$

$$\rho = \rho(p,s) \tag{41.04}$$

$$\operatorname{curl} (\vec{V} \times \vec{H}) = \vec{0} \tag{41.05}$$

where it is assumed that the flow is inviscid and thermally non-conducting.

For these flows, equations (41.02) and (41.05) reduce to

$$\rho(\vec{V} \cdot \operatorname{grad})\vec{V} + \operatorname{grad} \left(p + \frac{\mu H^2}{2} \right) = 0 \tag{41.06}$$

and

$$\operatorname{div} (H\vec{V}) = 0 \tag{41.07}$$

wherein $H = |\vec{H}|$.

From equations (41.01) and (41.07), we get

$$H = \lambda\ \rho \tag{41.08}$$

where λ is a scalar function satisfying

$$\vec{V}\ .\ \text{grad}\ \lambda = 0 \tag{41.09}$$

Substitution of (41.08) in (41.06) gives

$$\rho(\vec{V}\ .\ \text{grad})\vec{V} + \text{grad}\left(p + \mu\,\frac{\lambda^2\rho^2}{2}\right) = \vec{0} \tag{41.10}$$

These flows are studied by the help of six scalar equations (41.01), (41.03), (41.04), (41.08) and (41.10) in six dependent variables, the two components of the velocity vector $\vec{V}$, the scalar function H and the three thermodynamic variables p, ρ and s.

Section 2. Substitution Principle.

By use of the state equation, one of the thermodynamic variables can be eliminated, giving five equations in five dependent variables.

We now show that for an arbitrary equation of state, one more of the thermodynamic variables is redundant provided the pressure is constant along each streamline or the equation of state is of the product form. P. Smith (1963) has also worked towards a part of this problem.

Introducing a new vector $\vec{q} = \sqrt{\rho}\,\vec{V}$ in (41.01), (41.06) and (41.07), we find

$$\operatorname{div} \vec{q} + \frac{1}{2} (\vec{q} \cdot \operatorname{grad} \ln \rho) = 0 \tag{42.01}$$

$$(\vec{q} \cdot \operatorname{grad})\vec{q} - \frac{1}{2} (\vec{q} \cdot \operatorname{grad} \ln \rho)\vec{q} + \operatorname{grad} \left(p + \frac{\mu H^2}{2}\right) = \vec{0} \tag{42.02}$$

$$\operatorname{div} (H \vec{q}) - \frac{H}{2} (\vec{q} \cdot \operatorname{grad} \ln \rho) = 0 \tag{42.03}$$

wherein

$$\vec{q} \cdot \operatorname{grad} \ln \rho = \left[\frac{\partial}{\partial p} (\ln \rho) \right]_s \vec{q} \cdot \operatorname{grad} p \tag{42.04}$$

Equations (42.01) to (42.03) are four equations in five dependent variables $\vec{q}$, ρ, p and H. Equation (42.04) suggests that the number of dependent variables can be reduced to four provided either $\vec{q} \cdot \operatorname{grad} p = 0$ or $\left[\frac{\partial}{\partial p} (\ln \rho) \right]_s$ is a function of p alone.

The second condition implies that the equation of state is of the product form.

In the first case, the four equations in four unknowns are:

$$\operatorname{div} \vec{q} = 0 \tag{42.05}$$

$$(\vec{q} \cdot \operatorname{grad})\vec{q} + \operatorname{grad}\left(p + \frac{\mu H^2}{2}\right) = \vec{0} \tag{42.06}$$

$$\vec{q} \cdot \operatorname{grad} H = 0 \tag{42.07}$$

Therefore, we have:

<u>Theorem 4.1</u>. For conducting compressible fluid having an arbitrary equation of state, any plane transverse flow field satisfying the flow equations is a member of an infinite group of fields sharing the same streamlines, the same pressure field, the same magnetic field provided the pressure is constant on each streamline and the members of the group are related by $p^* = p$; $\rho^* = m^2\rho$; $\vec{V}^* = \frac{\vec{V}}{m}$; $H^* = H$; $\rho(p^*,s^*) = m^2\rho(p,s)$ where m is any scalar function constant on each individual streamline.

<u>Geometric Implications of Theorem 4.1</u>.

By assumption

$$\vec{V} \cdot \operatorname{grad} p = 0 \tag{42.08}$$

From (41.03), (41.04) and (42.08), we get

$$\vec{V} \cdot \operatorname{grad} \rho = 0 \tag{42.09}$$

Since ρ, p and λ are constant along each streamline, therefore, scalar multiplication of (41.10) with $\vec{V}$ yields

$$\vec{V} \cdot \operatorname{grad} V = 0$$

or

$$v_1^2 \frac{\partial v_1}{\partial x} + v_1 v_2 \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) + v_2^2 \frac{\partial v_2}{\partial y} = 0 \qquad (42.10)$$

Because of (42.09), (41.01) reduces to

$$\text{div } \vec{V} = 0$$

or

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \qquad (42.11)$$

Taking curl of (41.10) and using the fact that ρ is constant on each streamline, we obtain

$$\text{curl } [(\vec{V} \cdot \text{grad})\vec{V}] = \vec{0}$$

or

$$\frac{\partial}{\partial x}\left(v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y}\right) = \frac{\partial}{\partial y}\left(v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y}\right) \qquad (42.12)$$

Restrictions (42.10), (42.11) and (42.12) on velocity field imply:

<u>Theorem 4.2.</u> If the substitution principle holds for plane transverse flows with arbitrary equation of state then the only possible flow fields are the general vortex flows or flows in parallel straight lines.

Section 3. Three Theorems.

Let $\psi(x,y)$ = constant represent the streamlines and $\phi(x,y)$ = constant be their orthogonal trajectories, we find that the flow equations in natural (or streamline) co-ordinates are:

$$\frac{\partial}{\partial\phi}(\sqrt{G}\,\rho\, V) = 0 \tag{43.01}$$

$$\rho\, V\frac{\partial V}{\partial\phi} + \frac{\partial p}{\partial\phi} + \mu\,\lambda^2\,\frac{\partial\rho}{\partial\phi} = 0 \tag{43.02}$$

$$\frac{\rho V^2}{\sqrt{E}}\;\frac{\partial\sqrt{E}}{\partial\psi} - \frac{\partial p}{\partial\psi} - \mu\,\lambda\,\rho\,\frac{\partial}{\partial\psi}(\lambda\,\rho) = 0 \tag{43.03}$$

$$\frac{\partial s}{\partial\phi} = 0 \tag{43.04}$$

$$\frac{\partial\lambda}{\partial\phi} = 0 \tag{43.05}$$

wherein $\sqrt{E(\phi,\psi)}$ and $\sqrt{G(\phi,\psi)}$ are the metric coefficients of the net.

The squared element of arc length in this co-ordinate system is

$$ds^2 = E(\phi,\psi)\,d\phi^2 + G(\phi,\psi)\,d\psi^2$$

where the restriction of the (ϕ,ψ) net to a plane requires that E and G satisfy the Gauss's Equation

$$\frac{\partial}{\partial \phi}\left[\frac{1}{\sqrt{E}} \quad \frac{\partial \sqrt{G}}{\partial \phi}\right] + \frac{\partial}{\partial \psi}\left[\frac{1}{\sqrt{G}} \quad \frac{\partial \sqrt{E}}{\partial \psi}\right] = 0 \qquad (43.06)$$

Theorem 4.3. If the speed is sonic in plane transverse flows then the flow fields are the general vortex flows or flows in parallel straight lines.

Proof: By assumption

$$V = C(\rho, s) \qquad (43.07)$$

where C is the speed of sound.

Substitution of (43.07) in (43.02) yields

$$\left(\rho \ C \frac{\partial C}{\partial \rho} + C^2 + \mu \ \lambda^2 \ \rho\right) \frac{\partial \rho}{\partial \phi} = 0$$

The quantity $(\rho \ C \frac{\partial C}{\partial \rho} + C^2 + \mu \ \lambda^2 \ \rho)$ is strictly positive when $\frac{\partial^2 p}{\partial \rho^2} > 0$. Therefore, we have

$$\frac{\partial \rho}{\partial \phi} = 0$$

which implies

$$\frac{\partial p}{\partial \phi} = 0$$

Thus, the flow fields are general vortex flows or flows in parallel straight lines.

Theorem 4.4. If one of the four scalar quantities V, p, ρ or C is constant along each streamline, then the other three together with the vorticity ω are also constant along each streamline.

Proof: Because $p = p(\rho, s)$, $C = C(\rho, s)$ and $\frac{\partial s}{\partial \phi} = 0$, therefore, when one of the variables C, ρ or p is constant along each streamline, then the other two are also constant along each streamline. In section 2 of this chapter we have shown that $\frac{\partial p}{\partial \phi} = 0$ implies $\frac{\partial V}{\partial \phi} = 0$. In case $\frac{\partial V}{\partial \phi} = 0$, equation (43.02) requires that

$$(C^2 + \mu \lambda^2 \rho) \frac{\partial \rho}{\partial \phi} = 0$$

or

$$\frac{\partial \rho}{\partial \phi} = 0$$

Hence, if one of the four variables V, p, ρ or C is constant along each streamline, then the other three are also constant along each streamline. Finally, in plane flow, vorticity is given by

$$\omega = - \frac{1}{\sqrt{GE}} \frac{\partial}{\partial \psi} (\sqrt{E}\, V)$$

Therefore,

$$\frac{\partial \omega}{\partial \phi} = \frac{1}{G} \frac{\partial \sqrt{G}}{\partial \phi} \left[V \frac{\partial}{\partial \psi} (\ln\sqrt{E}) + \frac{\partial V}{\partial \psi} \right] - \frac{1}{\sqrt{G}} \left[\frac{\partial V}{\partial \phi} \frac{\partial}{\partial \psi} (\ln\sqrt{E}) + V \frac{\partial^2}{\partial \phi \partial \psi} (\ln\sqrt{E}) + \frac{\partial^2 V}{\partial \phi \partial \psi} \right]$$

and

$$\frac{\partial \omega}{\partial \psi} = \frac{1}{G} \frac{\partial \sqrt{G}}{\partial \psi} \left[V \frac{\partial}{\partial \psi} (\ln\sqrt{E}) + \frac{\partial V}{\partial \psi} \right] - \frac{1}{\sqrt{G}} \left[\frac{\partial V}{\partial \psi} \frac{\partial}{\partial \psi} (\ln\sqrt{E}) + V \frac{\partial^2}{\partial \psi^2} (\ln\sqrt{E}) + \frac{\partial^2 V}{\partial \psi^2} \right]$$

Obviously, if one of the scalar functions V, p, ρ or C is constant along each streamline, then

$$\frac{\partial \omega}{\partial \phi} = 0 \qquad \text{and} \qquad \frac{\partial \omega}{\partial \psi} \neq 0$$

Theorem 4.4 implies that the substitution principle is valid for plane transverse MFD flows provided the velocity magnitude or the pressure or the density or the local speed of sound is constant along each streamline.

Theorem 4.5. If the streamlines are concentric circles or parallel straight lines then the pressure is constant along each streamline.

Proof: In case of concentric circles

$$\sqrt{E} = \psi \qquad \text{and} \qquad \sqrt{G} = 1 \tag{43.08}$$

Using (43.08) in (43.01), we get

$$\frac{\partial \rho}{\partial \phi} + \frac{\rho}{V}\frac{\partial V}{\partial \phi} = 0 \tag{43.09}$$

Eliminating $\frac{\partial \rho}{\partial \phi}$ between (43.09) and (43.02), we obtain

$$(V^2 - C^2 - \mu\, \rho\, \lambda^2)\, \frac{\partial \rho}{\partial \phi} = 0$$

This equation implies that either $V^2 = C^2 + \mu\, \rho\, \lambda^2$ or $\frac{\partial \rho}{\partial \phi} = 0$. However, when $V^2 = C^2 + \mu\, \rho\, \lambda^2$, (43.09) implies that $\frac{\partial \rho}{\partial \phi} = 0$ when $\frac{\partial^2 p}{\partial \rho^2} > 0$.

Therefore, p is constant along each streamline.

If the streamlines are straight and parallel, we obtain the same result.

Section 4. Uniqueness of Incompressible Flows for Given Streamlines.

The condition of incompressibility reduces (41.01) and (41.09) to

$$\operatorname{div} \vec{V} = 0 \tag{44.01}$$

and

$$\vec{V} \cdot \operatorname{grad} H = 0 \tag{44.02}$$

Scalar multiplication of equation (41.06) with $\vec{V}$ and the condition (44.02) on H, yield

$$V^2 + \frac{2p}{\rho} = a^2 \tag{44.03}$$

where

$$\vec{V} \cdot \operatorname{grad} a = 0 \tag{44.04}$$

Taking curl of (41.06), we obtain

$$\operatorname{curl} [(\vec{V} \cdot \operatorname{grad})\vec{V}] = \vec{0} \tag{44.05}$$

Equations (44.01) and (44.05) are the necessary and sufficient conditons satisfied by $\vec{V}$. We now proceed to investigate under what conditions will the velocity field

$$\vec{V}^* = f(x,y)\vec{V}$$

satisfy the equations (44.01) and (44.05).

Equation (44.01) requires that

$$\vec{V} \cdot \operatorname{grad} f = 0 \tag{44.06}$$

From equation (44.05), we find that

$$2f[\text{grad } f \times (\vec{V} \cdot \text{grad})\vec{V}] + f^2 \text{ curl}[(\vec{V} \cdot \text{grad})\vec{V}]$$

$$+ \text{ curl}[f\, \vec{V}(\vec{V} \cdot \text{grad } f)] = \vec{0}$$

Using (44.05) and (44.06), we get

$$\text{grad } f \times (\vec{V} \cdot \text{grad})\vec{V} = \vec{0}$$

Making use of (41.06), we have

$$\text{grad } f \times \text{grad}\left(p + \frac{\mu H^2}{2}\right) = \vec{0} \qquad (44.07)$$

Because of (44.02), (44.03), (44.04) and (44.06), (44.07) reduces to

$$\text{grad } f \times \text{grad } V = \vec{0} \qquad (44.08)$$

The condition (44.08) has been discussed in section 4 of chapter III. It implies

<u>Theorem 4.6</u>. Any plane, incompressible, transverse MFD flow is unique unless it has a constant velocity magnitude along each individual streamline.

If we assume that the equation (43.04) holds, then (43.02) implies that the velocity magnitude is constant along each streamline. Therefore incompressible, plane, transverse MFD flows satisfying the equation (41.03) are not unique.

Section 5. Speed Equation.

To obtain the solutions of problems of steady, inviscid, transverse flows of infinitely conducting compressible fluids we consider the equations (43.01) to (43.05). From these equations, written in natural (or streamline) coordinates, we eliminate the three thermodynamical variables and the magnetic intensity and thus obtain a linear partial differential equation of the second order in speed.

From equation (43.01), we get

$$\sqrt{G}\ \rho\ V = A(\psi) \tag{45.01}$$

where $A(\psi)$ is an arbitrary function of ψ.

Eliminating ρ between (43.02), (45.01) and (43.03), (45.01), we have

$$\frac{A(\psi)}{\sqrt{G}} \frac{\partial V}{\partial \phi} + \frac{\partial p}{\partial \phi} + \mu \frac{\lambda^2 A^2(\psi)}{V\sqrt{G}} \frac{\partial}{\partial \phi}\left(\frac{1}{V\sqrt{G}}\right) = 0 \tag{45.02}$$

and

$$\frac{V\ A(\psi)}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial \psi} - \frac{\partial p}{\partial \psi} - \mu \frac{\lambda A(\psi)}{V\sqrt{G}} \frac{\partial}{\partial \psi}\left(\frac{\lambda A(\psi)}{V\sqrt{G}}\right) = 0 \tag{45.03}$$

Differentiating (45.02) with respect to ψ, (45.03) with respect to ϕ and adding, we get

$$\frac{\partial}{\partial \psi}\left[\frac{A(\psi)}{\sqrt{G}} \frac{\partial V}{\partial \phi} + \mu \frac{\lambda^2 A^2(\psi)}{V\sqrt{G}} \frac{\partial}{\partial \phi}\left(\frac{1}{V\sqrt{G}}\right)\right]$$
$$+ \frac{\partial}{\partial \phi}\left[- V\ A(\psi)\sqrt{\frac{E}{G}} \frac{\partial}{\partial \psi}\left(\frac{1}{\sqrt{E}}\right) - \mu \frac{\lambda\ A(\psi)}{V\sqrt{G}}\left\{\frac{1}{\sqrt{G}\ V} \frac{\partial}{\partial \psi}(\lambda\ A(\psi))\right.\right.$$
$$\left.\left. + \lambda\ A(\psi) \frac{\partial}{\partial \psi}\left(\frac{1}{V\sqrt{G}}\right)\right\}\right] = 0$$

or

$$\frac{A(\psi)}{\sqrt{G}} \frac{\partial^2 V}{\partial\phi\partial\psi} + \frac{1}{\sqrt{G}} A'(\psi) \frac{\partial V}{\partial\phi} + A(\psi) \frac{\partial V}{\partial\phi} \frac{\partial}{\partial\psi}\left(\frac{1}{\sqrt{G}}\right)$$

$$- V A(\psi)\sqrt{\frac{E}{G}} \frac{\partial^2}{\partial\phi\partial\psi}\left(\frac{1}{\sqrt{E}}\right) - V A(\psi)\frac{\partial}{\partial\phi}\left(\sqrt{\frac{E}{G}}\right)\frac{\partial}{\partial\psi}\left(\frac{1}{\sqrt{E}}\right)$$

$$- A(\psi) \sqrt{\frac{E}{G}} \frac{\partial V}{\partial\phi} \frac{\partial}{\partial\psi}\left(\frac{1}{\sqrt{E}}\right) = 0$$

The above equation can also be written as

$$\frac{\partial^2 V}{\partial\phi\partial\psi} + \left\{K(\psi) - \frac{\partial}{\partial\psi}\left(\ln\sqrt{\frac{G}{E}}\right)\right\}\frac{\partial V}{\partial\phi} + \sqrt{E}\left\{\frac{\partial}{\partial\psi}\left(\frac{1}{\sqrt{E}}\right)\frac{\partial}{\partial\phi}\left(\ln\sqrt{\frac{G}{E}}\right)\right.$$

$$\left. - \frac{\partial^2}{\partial\phi\partial\psi}\left(\frac{1}{\sqrt{E}}\right)\right\} V = 0 \tag{45.04}$$

where

$$K(\psi) = \frac{A'(\psi)}{A(\psi)} \tag{45.05}$$

Equation (45.04) is a linear hyperbolic partial differential equation of second order in canonical form in which V is the dependent variable and ϕ and ψ are the independent variables. Let

$$V(\phi,\psi) = g(\psi)\ U(\phi,\psi) \tag{45.06}$$

where $g(\psi)$ is an arbitrary function of ψ. According to the substitution principle $U(\phi,\psi)$ is the speed of a flow which has the same streamline pattern as that of our flow. Speed distributions given by the functions $U(\phi,\psi)$ and $V(\phi,\psi)$ are members of infinite set of possible speed distributions.

We take

$$g(\psi) = \exp\left\{-\int K(\psi)\, d\psi\right\} \tag{45.07}$$

Using (45.07), equation (45.04) takes the form

$$\frac{\partial^2 U}{\partial\phi\partial\psi} - \frac{\partial}{\partial\psi}\left(\ln\sqrt{\frac{G}{E}}\right)\frac{\partial U}{\partial\phi} + \sqrt{E}\left\{\frac{\partial}{\partial\psi}\left(\frac{1}{\sqrt{E}}\right)\frac{\partial}{\partial\phi}\left(\ln\sqrt{\frac{G}{E}}\right)\right.$$

$$\left. - \frac{\partial^2}{\partial\phi\partial\psi}\left(\frac{1}{\sqrt{E}}\right)\right\}V = 0 \tag{45.08}$$

This equation can be used to determine the velocity distribution of a flow problem.

Section 6. Density, Magnetic Field, Pressure, State Equation and Mach's Number.

Putting $V = U(\phi,\psi)$ in (45.01) and considering (45.05), (45.07), we get

$$\rho = \frac{N}{\sqrt{G}\; U} \tag{46.01}$$

where N is an arbitrary constant.

Using (46.01) in (41.08), we find that the magnetic field vector $\vec{H}$ is given by

$$\vec{H} = \left(o,\; o,\; \frac{\lambda N}{\sqrt{G}\; U} \right) \tag{46.02}$$

where λ is an arbitrary function of ψ which can be prescribed.

Substituting $V = U(\phi,\psi)$, (46.01) and (46.02) in (43.02) and (43.04) we obtain

$$\frac{\partial p}{\partial \phi} = -\frac{N}{\sqrt{G}} \frac{\partial U}{\partial \phi} - \mu \frac{\lambda^2 N^2}{\sqrt{G}\; U} \frac{\partial}{\partial \phi}\left(\frac{1}{\sqrt{G}\; U} \right) \tag{46.03}$$

and

$$\frac{\partial p}{\partial \psi} = \frac{NU}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial \psi} - \mu \frac{\lambda\; N^2}{\sqrt{G}\; U} \frac{\partial}{\partial \psi}\left(\frac{\lambda}{\sqrt{G}\; U} \right) \tag{46.04}$$

Equations (46.03), (46.04) and the fact that $p = p(\phi,\psi)$ gives us

$$p = p_o + \int_{\phi_o}^{\phi} -\left\{ \frac{N}{\sqrt{G}} \frac{\partial U}{\partial \phi} + \mu \frac{\lambda^2 N^2}{\sqrt{G}\; U} \frac{\partial}{\partial \phi}\left(\frac{1}{\sqrt{G}\; U} \right) \right\} d\phi$$

$$+ \int_{\psi_o}^{\psi} \left\{ \frac{NU}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial \psi} - \mu \frac{\lambda N^2}{\sqrt{G}\; U} \frac{\partial}{\partial \psi}\left(\frac{\lambda}{\sqrt{G}\; U} \right) \right\} d\psi \tag{46.05}$$

where ϕ_o, ψ_o and p_o denote arbitrary constants.

Suppose

$$\rho = \rho(\phi,\psi) \quad , \quad p = p(\phi,\psi) \tag{46.06}$$

are the density and pressure obtained in (46.01) and (46.05). On the other hand, since the specific entropy is constant on each streamline according to equation (43.04), we have

$$s = s(\psi) \tag{46.07}$$

Eliminating ϕ and ψ between the three equations (46.06) and (46.07) we find

$$\rho = \rho(p,s) \tag{46.08}$$

for the state equation.

Scalar multiplication of equation (41.10) with $\vec{V}'$ gives

$$\rho \vec{V} \cdot \text{grad} \frac{V^2}{2} + \vec{V} \cdot \text{grad} \left(p + \mu \frac{\lambda^2 \rho^2}{2} \right) = 0$$

or

$$\rho V^2 \frac{\partial V}{\partial \phi} + V \frac{\partial \rho}{\partial \phi} (C^2 + \mu \lambda^2 \rho) = 0 \tag{46.09}$$

since

$$C^2 = \frac{\partial p}{\partial \rho}$$

where C is the local speed of sound.

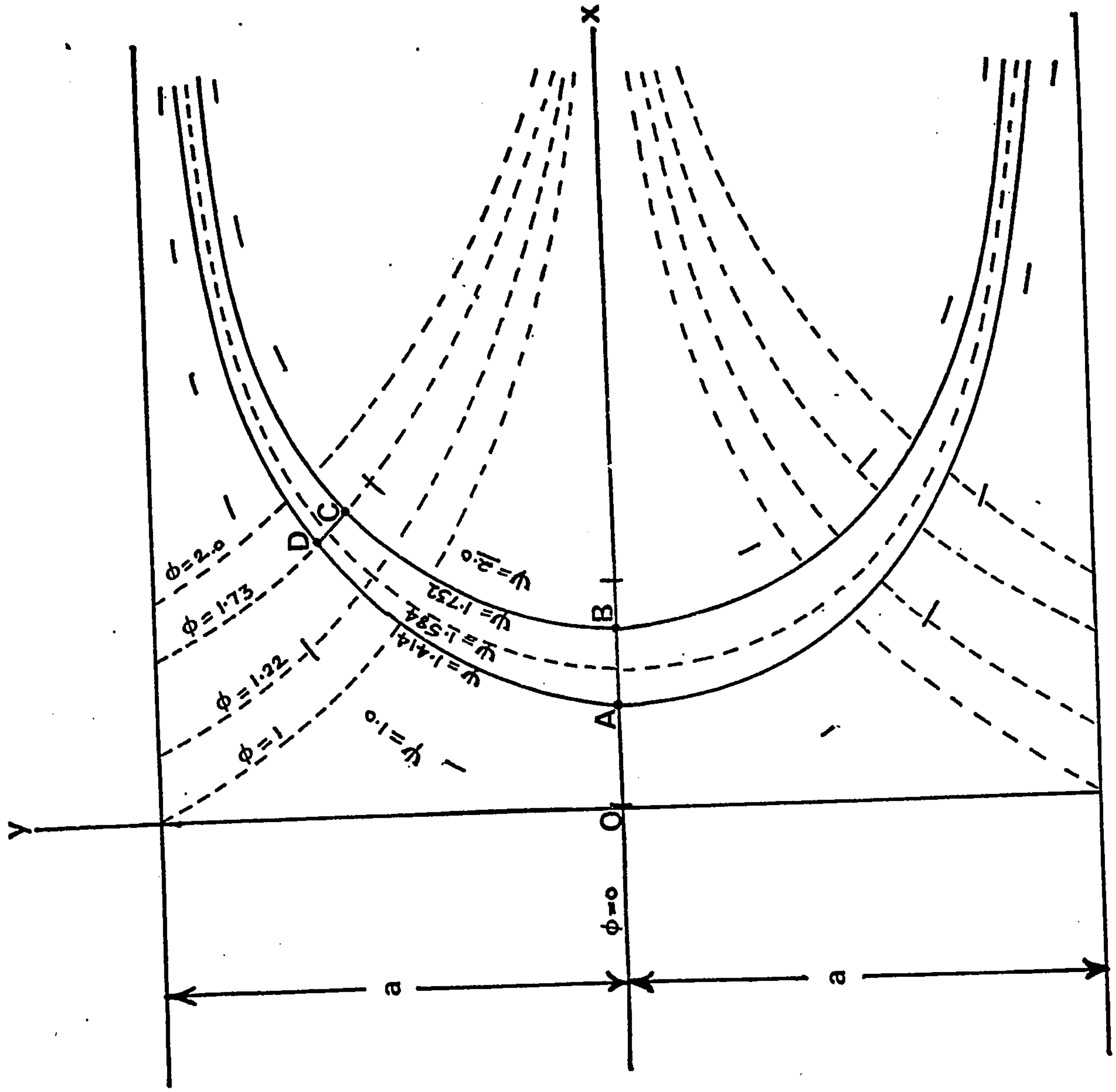
Using $V = U(\phi,\psi)$ and (46.01) in (46.09), we get

$$C^2 = \left[\frac{U^2 \frac{\partial U}{\partial \phi}}{\frac{U}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial \phi} + \frac{\partial U}{\partial \phi}} - \frac{\mu \lambda^2 N}{\sqrt{G}\, U} \right] \tag{46.10}$$

Substituting for C from (46.10) in $M = \frac{U}{C}$, we find Mach's number at any point.

In this way, according to the substitution principle, we find not only one solution but a family of infinite solutions given by speed $g(\psi)U$, density $\frac{\rho}{g^2(\psi)}$, magnetic field $\vec{H}$ and pressure p where ρ, $\vec{H}$ and p are given by (46.01), (46.02) and (46.05) respectively. All these solutions have the same streamlines.

Section 7. Flow in a Logrithmic Channel.



To study the flow through a logarithmic channel, we choose the logarithmic co-ordinate system as our natural (or streamline) co-ordinate system. For this curvilinear co-ordinate system, we have

$$\psi = \exp\left(\frac{\pi x}{2a}\right) \cos\left(\frac{\pi y}{2a}\right) \qquad 0 < \psi < \infty \qquad (47.01)$$

and

$$\phi = \exp\left(\frac{\pi x}{2a}\right) \sin\left(\frac{\pi y}{2a}\right) \qquad -\infty < \phi < \infty \qquad (47.02)$$

where ψ = constant are the streamlines and ϕ = constant are their orthogonal trajectories.

From (47.01) and (47.02), we get

$$\left.\begin{aligned} x &= \frac{a}{\pi} \ln(\phi^2 + \psi^2) \\ \text{and} \\ y &= \frac{2a}{\pi} \tan^{-1} \frac{\phi}{\psi} \end{aligned}\right\} \qquad (47.03)$$

We now study the flow in a channel whose walls are $\psi = \psi_1$ and $\psi = \psi_2$.

For the curvilinear co-ordinate system (47.03), the squared element of arc length is

$$ds^2 = \frac{4a^2}{\pi^2} \frac{1}{(\phi^2+\psi^2)} (d\phi^2 + d\psi^2) \qquad (47.04)$$

From (47.04), we find

$$\sqrt{E} = \sqrt{G} = \frac{2a}{\pi} \frac{1}{(\phi^2+\psi^2)^{1/2}}$$

where $\sqrt{E}$, $\sqrt{G}$ are the metric coefficients of the curvilinear net and E,G are given by (22.03).

Substituting these values of $\sqrt{E}$ and $\sqrt{G}$ in (45.08), we get

$$\frac{\partial^2 U}{\partial\phi\partial\psi} + \frac{\phi\psi}{(\phi^2+\psi^2)^2} U = 0 \tag{47.05}$$

To find the general solution of (47.05), we let

$$U = (\phi^2+\psi^2)^{1/2} W(\phi,\psi) \tag{47.06}$$

Second partial derivative of (47.06) with respect to ϕ and ψ is found to be

$$\frac{\partial^2 U}{\partial\phi\partial\psi} = (\phi^2+\psi^2)^{1/2} \frac{\partial^2 W}{\partial\phi\partial\psi} + \frac{\psi}{(\phi^2+\psi^2)^{1/2}} \frac{\partial W}{\partial\phi} + \frac{\phi}{(\phi^2+\psi^2)^{1/2}} \frac{\partial W}{\partial\psi}$$

$$- \frac{\phi\psi}{(\phi^2+\psi^2)^{3/2}} W \tag{47.07}$$

Using (47.06) and (47.07) in (47.05), we get

$$(\phi^2+\psi^2) \frac{\partial^2 W}{\partial\phi\partial\psi} + \psi \frac{\partial W}{\partial\phi} + \phi \frac{\partial W}{\partial\psi} = 0 \tag{47.08}$$

To simplify the above equation, we let

$$\left.\begin{aligned} &\xi = \phi^2 \\ \text{and} \\ &\eta = \psi^2 \end{aligned}\right\} \tag{47.09}$$

Substitution of (47.09) in (47.08) gives

$$(\xi+\eta)\,\frac{\partial^2 W}{\partial\xi\partial\eta} + 2\,\frac{\partial W}{\partial\xi} + 2\,\frac{\partial W}{\partial\eta} = 0 \tag{47.10}$$

This is the Poisson-Euler-Darboux [W.F. Ames (1965)] equation.

It can be solved as follows:

Let us consider the partial differential equation

$$(\xi+\eta)\,\frac{\partial^2 X}{\partial\xi\partial\eta} + 2\,\frac{\partial X}{\partial\xi} = 0 \tag{47.11}$$

Differentiating (47.11) twice with respect to ξ, we find

$$(\xi+\eta)\,\frac{\partial^2}{\partial\xi\partial\eta}\,\left(\frac{\partial^2 X}{\partial\xi^2}\right) + 2\,\frac{\partial}{\partial\xi}\,\left(\frac{\partial^2 X}{\partial\xi^2}\right) + 2\,\frac{\partial}{\partial\eta}\,\left(\frac{\partial^2 X}{\partial\xi^2}\right) = 0 \tag{47.12}$$

On comparing (47.10) and (47.12), we find that if we let

$$W = \frac{\partial^2 X}{\partial\xi^2} \tag{47.13}$$

then the two equations become identical.

Writing (47.11) as

$$\frac{\partial}{\partial\eta}\left\{(\xi+\eta)^2\,\frac{\partial X}{\partial\xi}\right\} = 0$$

we get

$$\frac{\partial X}{\partial\xi} = \frac{\gamma(\xi)}{(\xi+\eta)^2} \tag{47.14}$$

where $\gamma(\xi)$ is an arbitrary function of ξ.

From (47.13) and (47.14), we obtain

$$W = \frac{\partial}{\partial\xi}\left\{\frac{\gamma(\xi)}{(\xi+\eta)^2}\right\} \tag{47.15}$$

Similarly partial differentiation of the equation

$$(\xi+\eta)\ \frac{\partial^2 Y}{\partial\xi\partial\eta} + 2\ \frac{\partial Y}{\partial\eta} = 0 \tag{47.16}$$

twice with respect to η yields

$$(\xi+\eta)\ \frac{\partial^2}{\partial\xi\partial\eta}\ \left(\frac{\partial^2 Y}{\partial\eta^2}\right) + 2\ \frac{\partial}{\partial\xi}\ \left(\frac{\partial^2 Y}{\partial\eta^2}\right) + 2\ \frac{\partial}{\partial\eta}\ \left(\frac{\partial^2 Y}{\partial\eta^2}\right) = 0 \tag{47.17}$$

From (47.10), (47.16) and (47.17), we get

$$W = \frac{\partial}{\partial\eta}\left\{\frac{\delta(\eta)}{(\xi+\eta)^2}\right\} \tag{47.18}$$

where $\delta(\eta)$ is an arbitrary function of η.

Therefore the general solution of (47.10) is given by

$$W = K + \frac{\partial}{\partial\xi}\ \left\{\frac{\gamma(\xi)}{(\xi+\eta)^2}\right\} + \frac{\partial}{\partial\eta}\left\{\frac{\delta(\eta)}{(\xi+\eta)^2}\right\} \tag{47.19}$$

where K is an arbitrary constant.

On changing the independent variables from ξ,η to ϕ,ψ, we find that the general solution to equation (47.08) is

$$W = K + \frac{1}{2\phi}\ \frac{\partial}{\partial\phi}\left(\frac{\gamma_1(\phi)}{(\phi^2+\psi^2)^2}\right) + \frac{1}{2\psi}\ \frac{\partial}{\partial\psi}\left(\frac{\delta_1(\psi)}{(\phi^2+\psi^2)^2}\right) \tag{47.20}$$

where $\gamma_1(\phi)$ and $\delta_1(\psi)$ are arbitrary functions of ϕ and ψ respectively.

Substituting for W from (47.20) in (47.06), we get

$$U = (\phi^2+\psi^2)^{1/2}\left[K + \frac{1}{2\phi}\ \frac{\partial}{\partial\phi}\left(\frac{\gamma_1(\phi)}{(\phi^2+\psi^2)^2}\right) + \frac{1}{2\psi}\ \frac{\partial}{\partial\psi}\left(\frac{\delta_1(\psi)}{(\phi^2+\psi^2)^2}\right)\right] \tag{47.21}$$

which when put in (45.06) gives

$$V = g(\psi)\ (\phi^2+\psi^2)^{1/2}\left[K + \frac{1}{2\phi}\frac{\partial}{\partial\phi}\left(\frac{\gamma_1(\phi)}{(\phi^2+\psi^2)^2}\right) + \frac{1}{2\psi}\frac{\partial}{\partial\psi}\left(\frac{\delta_1(\psi)}{(\phi^2+\psi^2)^2}\right)\right]$$

Section 8. Boundary Value Problem

Equation (47.05) is a second order linear partial differential equation of hyperbolic type in canonical form. In this equation U is the dependent variable and ϕ, ψ are the independent variables. Our aim is to represent a solution U by properly prescribing the boundary value problem for it. For solving (47.05), to get speed at any point, we must prescribe the values of U and that of the 'outgoing' derivative of U on a curve C which is free curve (i.e. C is nowhere tangent to a characteristic). If, however, the initial curve degenerates into a right angle formed by the characteristics $\phi = C_1$, $\psi = C_2$, then we pose the boundary value problem called the characteristic boundary value problem in which merely the values of one quantity U on $\phi = C_1$ and $\psi = C_2$ be prescribed.

In the following, we solve (47.05) where U is prescribed along the wall $\psi = \psi_1$ and along an orthogonal trajectory $\phi = \phi_1$ such that

$$\left.\begin{aligned} U(\phi,\psi_1) &= \alpha(\phi) \qquad 0 \leq \phi \leq \phi_1 \\ U(\phi_1,\psi) &= \beta(\psi) \qquad \psi_1 \leq \psi \leq \psi_2 \end{aligned}\right\} \tag{48.01}$$

wherein $\beta(\psi_1) = \alpha(\phi_1)$.

Substituting

$$\left.\begin{aligned} r &= \phi^2 \\ s &= \psi^2 \end{aligned}\right\} \tag{48.02}$$

in (47.05), we get

$$\frac{\partial^2 U}{\partial r \partial s} + \frac{1}{4(r+s)^2} U = 0 \tag{48.03}$$

The boundary conditions (48.01) become

$$\left.\begin{aligned} U(r,s_1) &= a(r) & 0 \le r \le r_1 \\ U(r_1,s) &= b(s) & s_1 \le s \le s_2 \end{aligned}\right\} \tag{48.04}$$

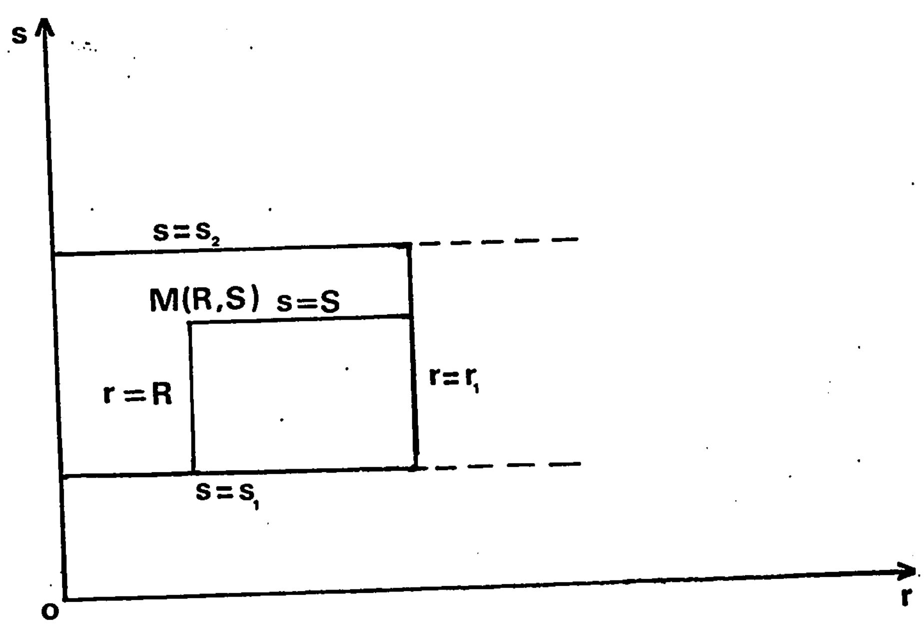
such that $a(r_1) = b(s_1)$

where $r_1 = \phi_1^2$, $s_1 = \psi_1^2$, $s_2 = \psi_2^2$.

We solve this problem by Riemann's method. Let $M(R,S)$ be any point inside the region given by

$$0 \le r \le r_1$$

$$s_1 \le s \le s_2.$$



The Riemann-Green function $A(r,s;\ R,S)$ for the equation (48.03) is the solution of the adjoint equation

$$\frac{\partial^2 A}{\partial r \partial s} + \frac{1}{4(r+s)^2} A = 0 \qquad (48.05)$$

such that

$$\left.\begin{array}{lll} & A(r,S;\ R,S) = 1 & \text{on } s = S \\ & A(R,s;\ R,S) = 1 & \text{on } r = R \\ \text{and} & A(R,S;\ R,S) = 1 & \text{at } (R,S) \end{array}\right\} \qquad (48.06)$$

From (48.06) we guess that $A(r,s;\ R,S)$ is of the form

$$A(r,s;\ R,S) = F(\alpha,\beta,\gamma;\ z) \qquad (48.07)$$

where

$$z = \frac{(r-R)\ (S-s)}{(R+S)\ (r+s)} \qquad (48.08)$$

so that conditions (48.06) hold. Here $F(\alpha,\beta,\gamma;\ z)$ is a hypergeometric function in which α,β and γ are unknown constants to be determined and

$$F(\alpha,\beta,\gamma;\ z) = 1 + \frac{\alpha\beta}{\gamma} z + \frac{\alpha(\alpha+1)\ \beta(\beta+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \ldots \qquad (48.09)$$

This series is convergent for $|z| < 1$ and is convergent for $|z| = 1$ also, if $\gamma - \alpha - \beta > 0$.

Substituting (48.07) in (48.05), we get

$$\frac{\partial^2 F(z)}{\partial z^2} \frac{\partial z}{\partial r} \frac{\partial z}{\partial s} + \frac{\partial^2 z}{\partial r \partial s} \frac{\partial F(z)}{\partial z} + \frac{1}{4(r+s)^2} F(z) = 0 \qquad (48.10)$$

From (48.08), we get

$$\left.\begin{aligned} \frac{\partial z}{\partial r}\ \frac{\partial z}{\partial s} &= \frac{1}{(r+s)^2}\,(z^2-z) \\ \frac{\partial^2 z}{\partial r \partial s} &= \frac{2z-1}{(r+s)^2} \end{aligned}\right\} \tag{48.11}$$

Substituting (48.11) in (48.10), we obtain

$$z(1-z)\ \frac{\partial^2 F(z)}{\partial z^2} + (1-2z)\ \frac{\partial F(z)}{\partial z} - \frac{1}{4}\,F(z) = 0 \tag{48.12}$$

This equation is the Gaussian differential equation with $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $\gamma = 1$. Therefore equation (48.12) possesses a unique solution $F(\frac{1}{2}, \frac{1}{2}, 1; z)$. Hence the Riemann-Green function is

$$A(r,s;\ R,S) = F(\tfrac{1}{2}, \tfrac{1}{2}, 1;\ z) \tag{48.13}$$

where z is given by (48.08). Therefore $F(\frac{1}{2}, \frac{1}{2}, 1; z)$ is only convergent for $|z| < 1$ as $\gamma - \alpha - \beta = 0$ here.

Therefore solution of U at M for (48.03) with the boundary conditions (48.04) is

$$\begin{aligned} U(M) = U(r_1,s_1)\ & F\left[\frac{1}{2}, \frac{1}{2}, 1;\ \frac{(r_1-R)\ (S-s_1)}{(r_1+s_1)\,(R+S)}\right] \\ & + \int_{r_1}^{R} F\left[\frac{1}{2}, \frac{1}{2}, 1;\ \frac{(r-R)\ (S-s_1)}{(r+s_1)\ (R+S)}\right] a'(r)\ dr \\ & + \int_{s_1}^{S} F\left[\frac{1}{2}, \frac{1}{2}, 1;\ \frac{(r_1-R)\,(S-s)}{(r_1+s)\,(R+S)}\right] b'(s)\ ds \end{aligned} \tag{48.14}$$

wherein the integrals on the right hand side (r,s) are the variables and (r_1, s_1) will be considered as constants. Equation (48.14) gives us the solution of the speed equation (47.05) with boundary conditions (48.01). Once U is determined, we find ρ, p, s and $\vec{H}$.

We now express the solution (48.14) in a form which will be more helpful in numerical calculations by finding a relationship between the hypergeometric function entering in our solution and the complete elliptic integral of the first type.

The complete elliptic integeral of the first type, denoted by K(k), is

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} .$$

or

$$K(k) = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + \ldots$$

or

$$K(k) = \frac{\pi}{2} F\left[\frac{1}{2}, \frac{1}{2}, 1; k^2\right] \tag{48.15}$$

Using (48.15) in (48.14), we find

$$U(M) = \frac{2}{\pi} U(r_1,s_1) \int_o^{\pi/2} \frac{d\phi}{\sqrt{1 - \frac{(r_1-R)(S-s_1)}{(r_1+s_1)\ (R+S)} \sin^2\phi}}$$

$$+ \frac{2}{\pi} \int_{r_1}^{R} \int_o^{\pi/2} a'(r) \frac{1}{\sqrt{1 - \frac{(r_1-R)\ (S-s_1)}{(r+s_1)\ (R+S)} \sin^2\phi}} d\phi\ dr$$

$$+ \frac{2}{\pi} \int_{s_1}^{S} \int_o^{\pi/2} b'(s) \frac{1}{\sqrt{1 - \frac{(r_1-R)\ (S-s)}{(r_1+s)\ (R+S)} \sin^2\phi}} d\phi\ ds$$

(48.16)

Section 9. A Numerical Example

In this section we will give some numerical results of an example. We choose

$$\psi = \psi_1 = \sqrt{2} \quad \text{and} \quad \psi = \psi_2 = \sqrt{3} \tag{49.01}$$

as the walls of the channel.

We suppose that the speed on the arcs $\psi = \sqrt{2}$ and $\phi = \sqrt{3}$ is respectively given by

$$\left.\begin{array}{ll} U = 5\phi^2 + 3 & \sqrt{2} \leq \psi \leq \sqrt{3} \\ \text{and} & \\ U = 7\psi^2 + 4 & 0 \leq \phi \leq \sqrt{3} \end{array}\right\} \tag{49.02}$$

On solving (47.05) with boundary conditions (49.02) by the method of finite-difference approximations, we find that speed distribution in the region

$$\sqrt{2} \leq \psi \leq \sqrt{3}$$

and

$$0 \leq \phi \leq \sqrt{3}$$

is given by the following table

ϕ	ψ	U
0.0	1.449	3.637
	1.483	4.272
	1.517	4.907
	1.549	5.541
	1.581	6.176
	1.612	6.810
	1.643	7.444
	1.673	8.078
	1.703	8.712

ϕ	ψ	U
0.548	1.449	5.143
	1.483	5.785
	1.517	6.427
	1.549	7.069
	1.581	7.710
	1.612	8.352
	1.643	8.994
	1.673	9.636
	1.703	10.278
	1.732	10.920
0.775	1.449	6.650
	1.483	7.299
	1.517	7.948
	1.549	8.597
	1.581	9.246
	1.612	9.895
	1.643	10.544
	1.673	11.193
	1.703	11.843
	1.732	12.493
0.949	1.449	8.156
	1.483	8.813
	1.517	9.469
	1.549	10.125
	1.581	10.781
	1.612	11.437
	1.643	12.094
	1.673	12.750
	1.703	13.407
	1.732	14.064

ϕ	ψ	U
1.095	1.449	9.663
	1.483	10.326
	1.517	10.989
	1.549	11.653
	1.581	12.316
	1.612	12.979
	1.643	13.642
	1.673	14.306
	1.703	14.970
	1.732	15.634
1.225	1.449	11.170
	1.483	11.840
	1.517	12.509
	1.549	13.179
	1.581	13.849
	1.612	14.519
	1.643	15.189
	1.673	15.860
	1.703	16.530
	1.732	17.201
1.342	1.449	12.676
	1.483	13.353
	1.517	14.029
	1.549	14.705
	1.581	15.381
	1.612	16.058
	1.643	16.735
	1.673	17.412
	1.703	18.088
	1.732	18.765

φ	ψ	U
1.449	1.449	14.183
	1.483	14.865
	1.517	15.548
	1.549	16.230
	1.581	16.913
	1.612	17.596
	1.643	18.179
	1.673	18.961
	1.703	19.644
	1.732	20.327
1.549	1.449	15.688
	1.483	16.377
	1.517	17.066
	1.549	17.754
	1.581	18.443
	1.612	19.132
	1.643	19.821
	1.673	20.509
	1.703	21.198
	1.732	21.887
1.643	1.449	17.194
	1.483	17.889
	1.517	18.583
	1.549	19.278
	1.581	19.972
	1.612	20.667
	1.643	21.136
	1.673	22.056
	1.703	22.750
	1.732	23.445

Using (49.01) and (49.02) in (48.16), we find

$$U(M) = \frac{2}{\pi}\left[18 \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \frac{(3-R)\ (S-2)}{5(R+S)}\sin^2\phi}} - 5\int_R^3 \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \frac{(r-R)}{(r+2)}\frac{(S-2)}{(R+S)}\sin^2\phi}}\, dr + 7\int_2^S \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \frac{(3-R)}{(3+s)}\frac{(S-s)}{(R+S)}\sin^2\phi}}\, ds \right] \tag{49.03}$$

On solving (49.03) for different pts. M in the region

$$\sqrt{2} \le \psi \le \sqrt{3}$$

$$0 \le \phi \le \sqrt{3}$$

we find that the speed distribution is given by the following tables and graphs:

ϕ	ψ	U
0.0	1.449	3.763
	1.483	4.528
	1.517	5.293
	1.549	6.058
	1.581	6.824
	1.612	7.590
	1.643	8.137
	1.673	9.123
	1.703	9.889
	1.732	10.655

φ	ψ	U
0.548	1.449	5.257
	1.483	6.014
	1.517	6.772
	1.549	7.530
	1.581	8.288
	1.612	9.046
	1.643	9.804
	1.673	10.562
	1.703	11.320
	1.732	12.078
0.775	1.449	6.750
	1.483	7.500
	1.517	8.250
	1.549	9.000
	1.581	9.751
	1.612	10.502
	1.643	11.252
	1.673	12.003
	1.703	12.753
	1.732	13.503
0.949	1.449	8.243
	1.483	8.986
	1.517	9.729
	1.549	10.473
	1.581	11.216
	1.612	11.959
	1.643	12.702
	1.673	13.445
	1.703	14.188
	1.732	14.931

ϕ	ψ	U
1.095	1.449	9.736
	1.483	10.472
	1.517	11.209
	1.549	11.945
	1.581	12.681
	1.612	13.418
	1.643	14.154
	1.673	14.890
	1.703	15.626
	1.732	16.361
1.225	1.449	11.230
	1.483	11.960
	1.517	12.689
	1.549	13.419
	1.581	14.148
	1.612	14.878
	1.643	15.607
	1.673	16.337
	1.703	17.066
	1.732	17.779
1.342	1.449	12.723
	1.483	13.446
	1.517	14.170
	1.549	14.893
	1.581	15.616
	1.612	16.339
	1.643	17.062
	1.673	17.785
	1.703	18.508
	1.732	19.231

φ	ψ	U
1.449	1.449	14.217
	1.483	14.934
	1.517	15.651
	1.549	15.368
	1.581	17.085
	1.612	17.802
	1.643	18.519
	1.673	19.236
	1.703	19.953
	1.732	20.670
1.549	1.449	15.711
	1.483	16.422
	1.517	17.114
	1.549	17.845
	1.581	18.556
	1.612	19.227
	1.643	19.978
	1.673	20.689
	1.703	21.40
	1.732	22.111
1.643	1.449	17.205
	1.483	17.911
	1.517	18.616
	1.549	19.322
	1.581	20.027
	1.612	20.733
	1.643	21.438
	1.673	22.143
	1.703	22.849
	1.732	23.554

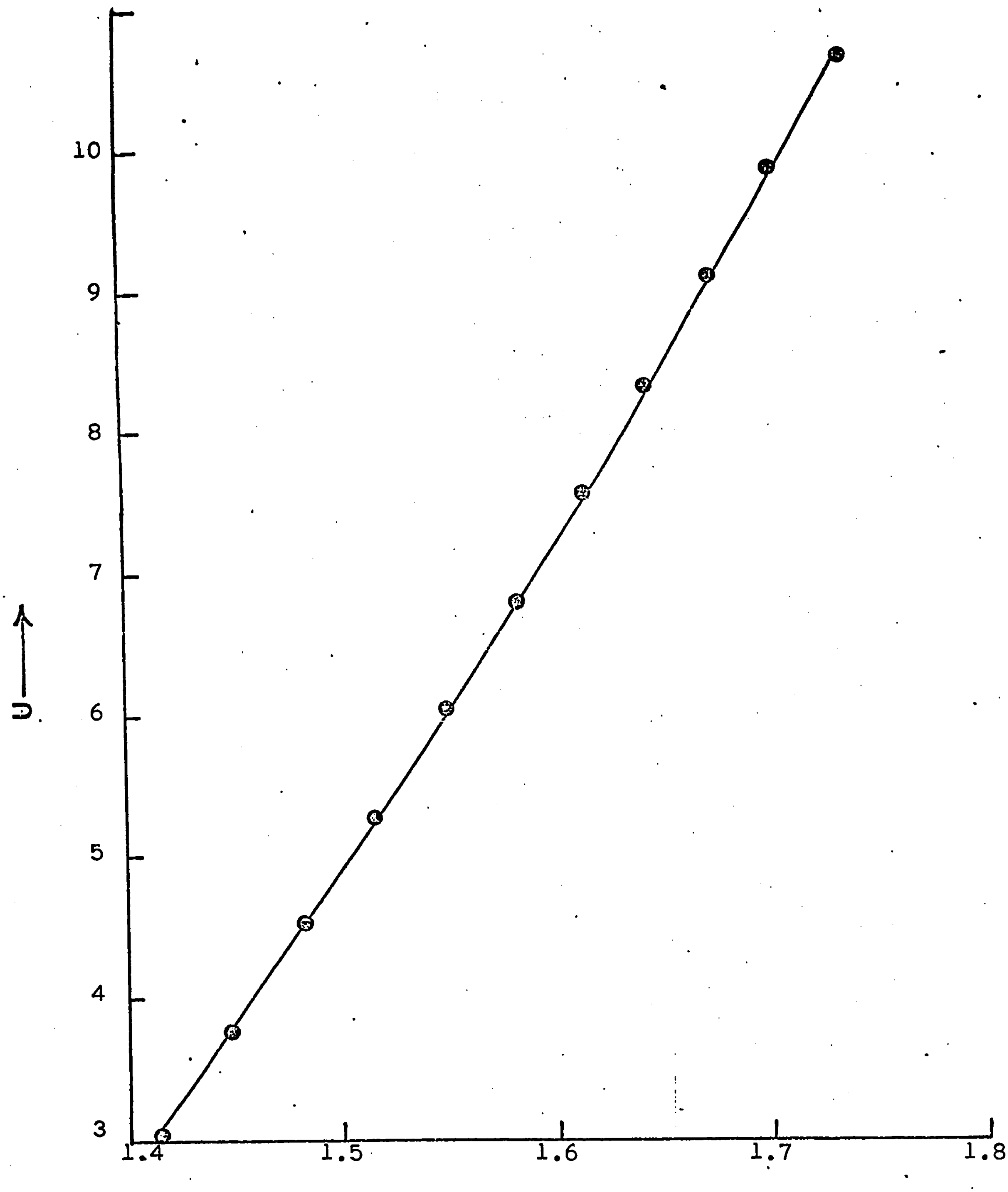


FIG.I VARIATION OF SPEED ON SECTION A B

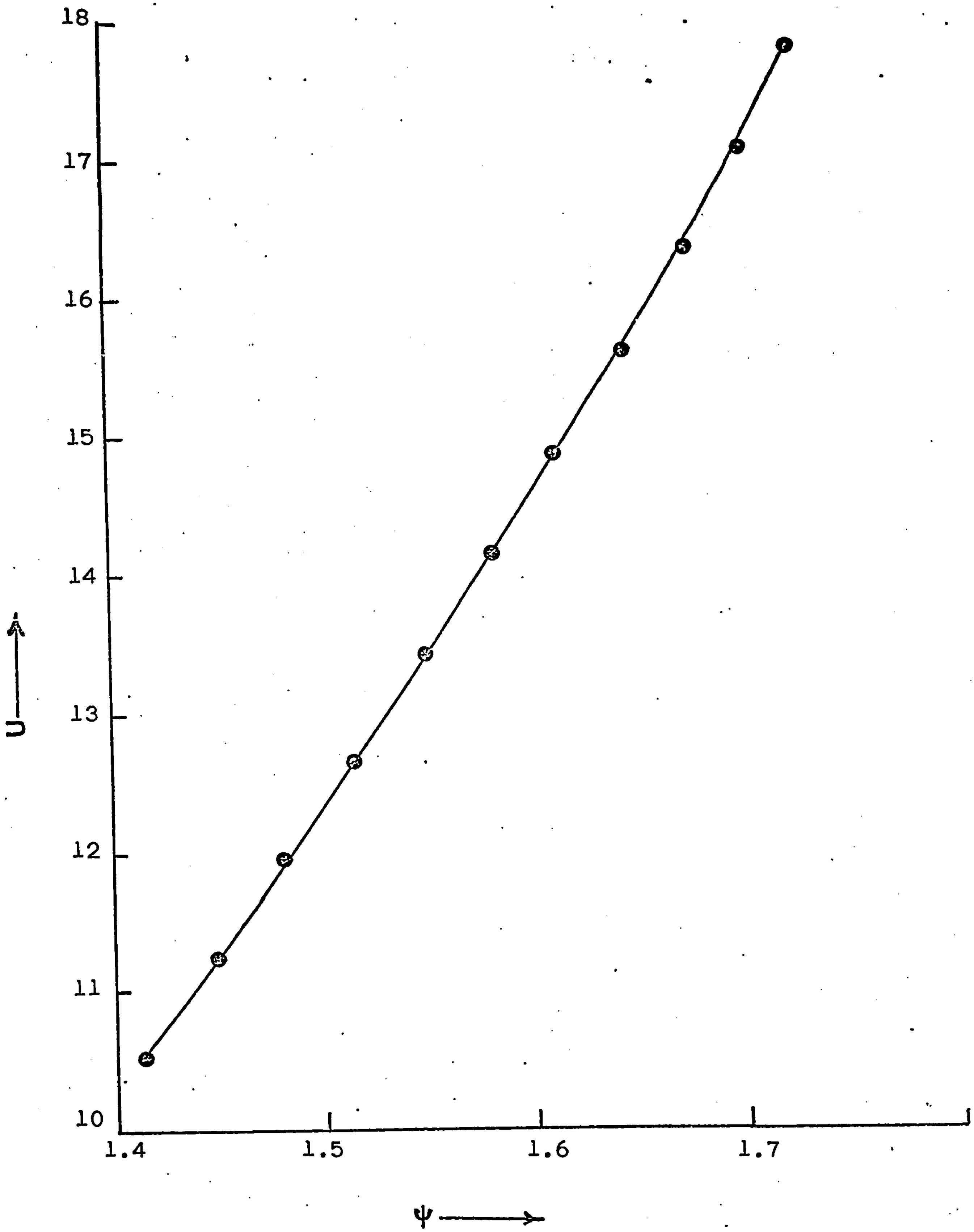


FIG.2 VARIATION OF SPEED ON THE ORTHOGONAL

TRAJECTORY ϕ = 1.225

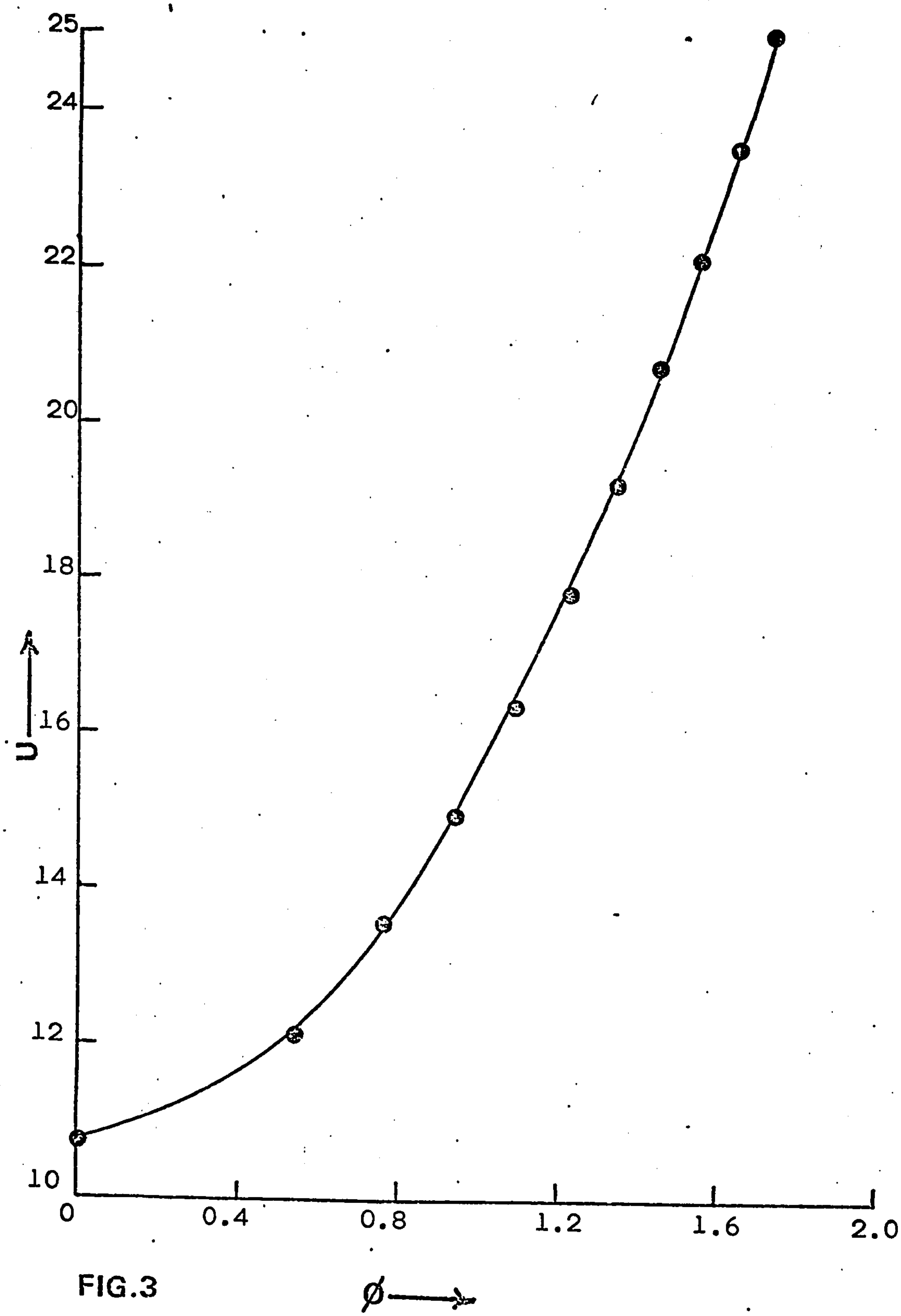


FIG.3

VARIATION OF SPEED ON THE WALL

$\psi = 1.732$

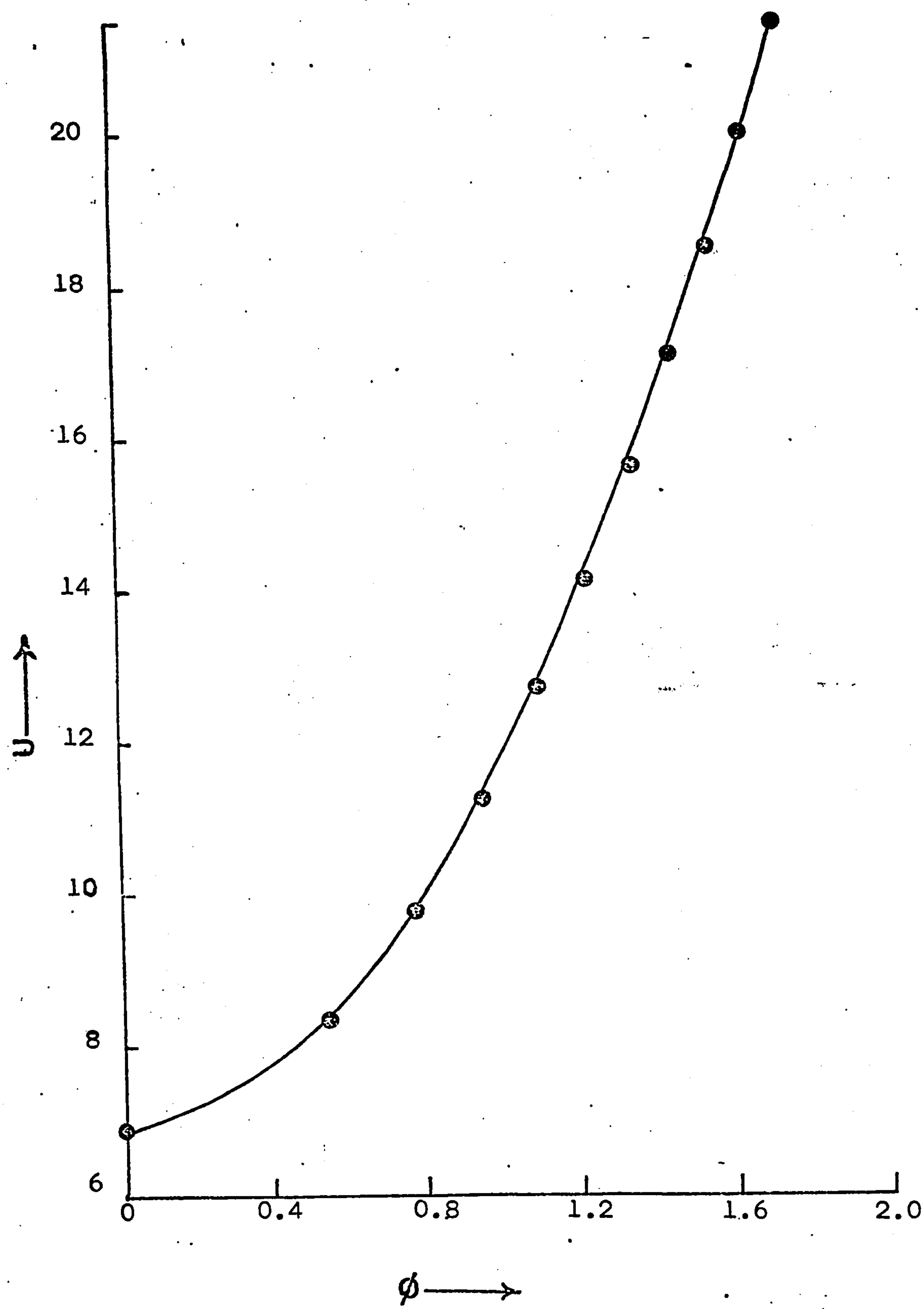


FIG. 4 VARIATION OF SPEED ON THE STREAMLINE

$\psi = 1.58$

CHAPTER V

ON PLANE MFD FLOWS WITH ORTHOGONAL MAGNETIC AND VELOCITY FIELD DISTRIBUTIONS.

Section 1. Flow Equations.

Steady, thermally non-conducting, inviscid and electrically infinitely conducting flows are governed by:

$$\operatorname{div} (\rho \vec{V}) = 0 \tag{51.01}$$

$$\rho(\vec{V} \cdot \operatorname{grad})\vec{V} + \operatorname{grad} p = \mu(\operatorname{curl} \vec{H}) \times \vec{H} \tag{51.02}$$

$$\vec{V} \cdot \operatorname{grad} s = 0 \tag{51.03}$$

$$\rho = \rho(p,s) \tag{51.04}$$

$$\operatorname{curl}(\vec{V} \times \vec{H}) = \vec{0} \tag{51.05}$$

$$\operatorname{div} \vec{H} = 0 \tag{51.06}$$

For two dimensional flows, with $\vec{H}$ in the plane of flow and $\vec{V} \cdot \vec{H} = 0$, we have

$$\vec{V} = (v_1, v_2) \qquad \text{and} \qquad \vec{H} = (-K v_2, K v_1) \tag{51.07}$$

where K is a scalar function.

Equation (51.05), on using (51.07), gives

$$HV = \lambda \quad \text{and} \quad \vec{H} = \left(- \frac{\lambda v_2}{v_1{}^2+v_2^2} , \frac{\lambda v_1}{v_1^2+v_2^2} \right) \tag{51.08}$$

where λ is an arbitrary constant.

Let ψ = constant be the streamlines and ϕ = constant their orthogonal trajectories i.e. the magnetic lines. In the curvilinear co-ordinate system (ϕ,ψ) the square of the element of arc length is given by

$$ds^2 = E\, d\phi^2 + G\, d\psi^2$$

where E and G are given by (22.03). The equation (22.16) becomes

$$\frac{\partial}{\partial\phi}\left\{\frac{1}{\sqrt{E}}\ \frac{\partial\sqrt{G}}{\partial\phi}\right\} + \frac{\partial}{\partial\psi}\left\{\frac{1}{\sqrt{G}}\ \frac{\partial\sqrt{E}}{\partial\psi}\right\} = 0 \tag{51.09}$$

In this co-ordinate system, equations (51.01), (51.02), (51.03), (51.04) and (51.06) read

$$\frac{\partial}{\partial\phi}\ (\sqrt{G}\ \rho\ V) = 0 \tag{51.10}$$

$$V\ \frac{\partial V}{\partial\phi} + \frac{1}{\rho}\ \frac{\partial p}{\partial\phi} + \frac{\lambda^2\mu}{\rho V\sqrt{G}}\ \frac{\partial}{\partial\phi}\left(\frac{\sqrt{G}}{V}\right) = 0 \tag{51.11}$$

$$\frac{V^2}{\sqrt{E}}\ \frac{\partial\sqrt{E}}{\partial\psi} - \frac{1}{\rho}\ \frac{\partial p}{\partial\psi} = 0 \tag{51.12}$$

$$\frac{\partial s}{\partial\phi} = 0 \tag{51.13}$$

$$\rho = \rho(p,s) \tag{51.04}$$

$$\frac{\partial}{\partial\psi}\left(\frac{\sqrt{E}}{V}\right) = 0 \tag{51.14}$$

Section 2. General Theorems.

Theorem 5.1. If the velocity magnitude is constant along each streamline in plane flow and $\lambda^2\mu \neq \rho\, C^2V^2$ in flow region, then the only possible flow fields are either general vortex flows or flows in parallel straight lines.

Proof: Since the speed is constant along each streamline, we have

$$V = V(\psi) \tag{52.01}$$

Using (52.01) in (51.14), we get

$$\sqrt{E} = A(\phi)\; V(\psi) \tag{52.02}$$

where $A(\phi)$ is an arbitrary function of ϕ.

From (52.01) and (51.10), we have

$$\frac{\partial \rho}{\partial \phi} + \frac{\rho}{\sqrt{G}}\;\frac{\partial \sqrt{G}}{\partial \phi} = 0 \tag{52.03}$$

Equations (51.11) and (52.01) give us

$$\frac{\partial p}{\partial \phi} + \frac{\lambda^2\mu}{V^2\sqrt{G}}\;\frac{\partial \sqrt{G}}{\partial \phi} = 0 \tag{52.04}$$

Because of (51.13), equation (52.04) can be written as

$$C^2\;\frac{\partial \rho}{\partial \phi} + \frac{\lambda^2\mu}{V^2\;\sqrt{G}}\;\;\frac{\partial \sqrt{G}}{\partial \phi} = 0 \tag{52.05}$$

where C is the speed of sound in the fluid. Eliminating $\frac{\partial \rho}{\partial \phi}$ between (52.03) and (52.05), we obtain

$$(\lambda^2\mu - \rho\; V^2\; C^2)\;\frac{\partial \sqrt{G}}{\partial \phi} = 0 \tag{52.06}$$

When $\lambda^2\mu \neq \rho\, V^2C^2$ in flow region, (52.06) implies

$$\sqrt{G} = g(\psi) \tag{52.07}$$

where $g(\psi)$ is an arbitrary function of ψ.

Substituting for $\sqrt{E}$ and $\sqrt{G}$ from (52.02) and (52.07) in (51.09), we find

$$V'(\psi) = C_1\, g(\psi)$$

or

$$V(\psi) = C_1 \int g(\psi)\, d\psi + C_2$$

where C_1 and C_2 are arbitrary constants.

It, therefore, follows that

$$\left.\begin{aligned} \sqrt{E} &= A(\phi)\left[C_1 \int g(\psi)\, d\psi + C_2\right] \\ \text{and} \quad \sqrt{G} &= g(\psi) \end{aligned}\right\} \tag{52.08}$$

Introducing the ordinary complex variable $z = x + iy$ and by equations (22.03), we have

$$E = \left|\frac{\partial z}{\partial \phi}\right|^2 \quad \text{and} \quad G = \left|\frac{\partial z}{\partial \psi}\right|^2$$

or

$$\frac{\partial z}{\partial \phi} = \sqrt{E}\, e^{i\alpha} \quad \text{and} \quad \frac{\partial z}{\partial \psi} = \sqrt{G}\, e^{i\beta} \tag{52.09}$$

where α and β, real functions of ϕ and ψ, are respectively the angles made by a streamline and an orthogonal trajectory with x-axis.

Substituting for $\frac{\partial x}{\partial \phi}$, $\frac{\partial y}{\partial \phi}$, $\frac{\partial x}{\partial \psi}$, $\frac{\partial y}{\partial \psi}$ from (52.09) in the second equation of (22.03), we get

$$\cos(\alpha - \beta) = 0$$

which implies that

$$\sin(\alpha - \beta) = \pm 1$$

Hence, we see that

$$e^{i\beta} = e^{i\alpha} e^{i(\beta-\alpha)} = \pm i e^{i\alpha}$$

Without loss of generality, we take

$$e^{i\beta} = i e^{i\alpha} \tag{52.10}$$

Eliminating z between the equations of (52.09) and using (52.10), we get

$$\frac{\partial}{\partial \psi} (\sqrt{E} e^{i\alpha}) - i \frac{\partial}{\partial \phi} (\sqrt{G} e^{i\alpha}) = 0$$

Separating into real and imaginary parts, we get

$$\left.\begin{aligned} \frac{\partial \alpha}{\partial \phi} &= - \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial \psi} \\ \frac{\partial \alpha}{\partial \psi} &= \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial \phi} \end{aligned}\right\} \tag{52.11}$$

From equations (52.08) and (52.11), we find

$$\alpha = \alpha(\phi) \quad \text{and} \quad \alpha'(\phi) = - C A(\phi) \tag{52.12}$$

Putting (52.08) and (52.12) in (52.09), we obtain

$$\left.\begin{aligned} &\frac{\partial z}{\partial \phi} = - \alpha'(\phi) \left[\int g(\psi)\, d\psi + \frac{C_2}{C_1} \right] (\cos\alpha + i \sin\alpha) \\ &\text{and} \\ &\frac{\partial z}{\partial \psi} = i\, g(\psi) (\cos\alpha + i \sin\alpha) \end{aligned}\right\} \tag{52.13}$$

when $C_1 \neq 0$.

In case $C_1 = 0$, equations (52.12) require that

$$\alpha = \text{constant}$$

i.e. the streamlines are parallel straight lines.

Solving (52.13) for z, we get

$$z = i\left[\int g(\psi)\,d\psi + \frac{C_2}{C_1}\right](\cos\alpha + i\sin\alpha) + K \qquad (52.14)$$

where K is an arbitrary constant.

By separating (52.14) into real and imaginary parts, we find

$$\left.\begin{aligned} x &= K_1 - \sin\alpha(\phi)\left[\int g(\psi)\,d\psi + \frac{C_2}{C_1}\right] \\ \text{and} \quad y &= K_2 + \cos\alpha(\phi)\left[\int g(\psi)\,d\psi + \frac{C_2}{C_1}\right] \end{aligned}\right\} \qquad (52.15)$$

where $K = K_1 + i\,K_2$.

Eliminating ϕ and ψ respectively between the two equations of (52.15), we obtain

$$(x - K_1)^2 + (y - K_2)^2 = \left[\int g(\psi)\,d\psi + \frac{C_2}{C_1}\right]^2 \qquad (52.16)$$

and

$$\frac{y - K_2}{x - K_1} = -\cot\alpha(\phi) \qquad (52.17)$$

Equation (52.16) implies that the streamlines are concentric circles.

<u>Corollary</u>. If the velocity magnitude is constant on each streamline and the fluid is polytropic gas, then the streamlines are concentric circles or parallel straight lines.

By assumption, equation (51.04) is

$$p = N(s)\ \rho^{\gamma}$$

where N(s) is an arbitrary function of s and γ is a constant.

From equation (52.06), for this state equation, we have

$$\sqrt{G} = g(\psi) \quad \text{or} \quad \rho = \left[\frac{\lambda^2\mu}{V^2\gamma N(s)}\right]^{1/\gamma} = \rho(\psi)$$

as both s and V are constant along each streamline. However, $\rho = \rho(\psi)$ again implies, using (51.10) that

$$\sqrt{G} = g(\psi)$$

This metric restriction along with metric restriction $\sqrt{E} = A(\phi)\left[C_1 \int g(\psi)\ d\psi + C_2\right]$ obtained in (52.08) prove the stated geometric implication.

<u>Theorem 5.2.</u> If the plane flow is sonic and

$$\lambda^2\mu \neq \rho\ C^4\ \frac{\frac{\partial C}{\partial \rho} + \frac{C}{\rho}}{2\frac{\partial C}{\partial \rho} + \frac{C}{\rho}}$$ in flow region, then the only possible flow fields are the general vortex flows or flows in parallel straight lines.

<u>Proof</u>: By assumption

$$V = C(\rho, s) \tag{52.18}$$

where C is the speed of sound. Using this assumption in (51.14), we get

$$\sqrt{E} = A(\phi)\ C \tag{52.19}$$

Equations (52.18), (51.13) and (51.10) give

$$\frac{\partial \sqrt{G}}{\partial \phi} = - \frac{\sqrt{G}}{\rho C}\left\{\rho \frac{\partial C}{\partial \rho} + C\right\}\frac{\partial \rho}{\partial \phi} \tag{52.20}$$

Using (51.13) and (52.20), (51.11) becomes

$$\left[C \frac{\partial C}{\partial \rho} + \frac{C^2}{\rho} - \frac{\lambda^2 \mu}{\rho^2 C^3}\left\{2 \frac{\partial C}{\partial \rho} + C\right\}\right]\frac{\partial \rho}{\partial \phi} = 0$$

From this equation, if $\lambda^2\mu \neq \rho\, C^4 \dfrac{\frac{\partial C}{\partial \rho} + \frac{C}{\rho}}{2\frac{\partial C}{\partial \rho} + \frac{C}{\rho}}$ in the flow region, then we have

$$\rho = \rho(\psi) \tag{52.21}$$

and

$$V = C = C(\rho, s) = C(\psi) \tag{52.22}$$

Substituting (52.21) and (52.22) in (51.10) and (52.19), we obtain

$$\sqrt{G} = g(\psi)$$

and

$$\sqrt{E} = A(\phi)\ C(\psi)$$

These forms for $\sqrt{G}$ and $\sqrt{E}$ are the same as in the previous theorem.

Hence the required result.

Theorem 5.3. If one of the variables C, ρ or p is constant along each streamline, then the other two flow variables, the velocity magnitude, the vorticity are also constant along each streamline and the flow fields are the general vortex flows or flows in parallel straight lines.

Proof: Because $p = p(\rho, s)$, $C = C(\rho, s)$ and $\frac{\partial s}{\partial \phi} = 0$, therefore, if one of the three variables C, ρ or p is constant along each streamline, then the other two are also constant along each streamline. Employing this characteristic in (51.10) and (51.11), we find

$$\frac{\partial \sqrt{G}}{\partial \phi} + \frac{\sqrt{G}}{V} \frac{\partial V}{\partial \phi} = 0 \tag{52.23}$$

and

$$V \frac{\partial V}{\partial \phi} + \frac{\lambda^2 \mu}{\rho V^2 \sqrt{G}} \left[\frac{\partial \sqrt{G}}{\partial \phi} - \frac{\sqrt{G}}{V} \frac{\partial V}{\partial \phi} \right] = 0 \tag{52.24}$$

Elimination of $\sqrt{G}$ between equations (52.23) and (52.24) yields

$$\frac{\partial V}{\partial \phi} = 0 \qquad \text{or} \qquad V = \left(2\frac{\lambda^2 \mu}{\rho} \right)^{1/4}$$

that is, the velocity magnitude is constant along each streamline. Therefore, theorem 5.1 implies that the streamlines are either concentric circles or parallel straight lines.

As in section 3 of Chapter IV, it can be verified that for the flows discussed above

$$\frac{\partial \omega}{\partial \phi} = 0$$

This proves the stated theorem.

We now study the possible plane flow patterns of a conducting gas with infinite electrical conductivity which have the property that the streamlines and their orthogonal trajectories form an isometric net and the pressure is constant on each orthogonal trajectory. These assumptions give

$$\left.\begin{aligned} &E(\phi,\psi) = G(\phi,\psi) = H^2(\phi,\psi) \text{ (say)} \\ &\text{and} \\ &p(\phi,\psi) = p(\phi) \end{aligned}\right\} \quad (52.25)$$

Assumption (52.25) reduces the equation (51.12) to

$$H = H(\phi) \quad (52.26)$$

The conditions (51.09) and (52.26) imply that

$$H = e^{B\phi+D} \quad (52.27)$$

where B and D are arbitrary constants.

Substitution of (52.27) in (52.11) yields

$$\alpha = B\psi + E \quad (52.28)$$

where E is an arbitrary constant.

Use of (52.27) and (52.28) into (52.09) leads to

$$\left.\begin{aligned} &\frac{\partial z}{\partial \phi} = e^{B\phi+D}\, e^{i(B\psi+E)} \\ &\text{and} \\ &\frac{\partial z}{\partial \psi} = i\, e^{B\phi+D}\, e^{i(B\psi+E)} \end{aligned}\right\} \quad (52.29)$$

Solving (52.29) for z, we get

$$z = \frac{1}{B}\, e^{B\phi+D}\, e^{i(B\psi+E)} + M$$

where M is an aribtrary constant. Separating the real and imaginary parts, we get

$$x = \frac{1}{B}\, e^{B\phi+D} \cos(B\psi + E) + M_1$$

and

$$y = \frac{1}{B}\, e^{B\phi+D} \sin(B\psi + E) + M_2$$

where $M = M_1 + i\,M_2$. Elimination of ϕ and ψ from these two equations give

$$\frac{y - M_2}{x - M_1} = \tan(B\psi + E) \tag{52.30}$$

and

$$(x - M_1)^2 + (y - M_2)^2 = \frac{1}{B^2}\, e^{2(B\phi+D)} \tag{52.31}$$

From (52.30) and (52.31), we conclude that when $B \neq 0$ the streamlines are straight lines through the point (M_1, M_2) and the magnetic lines are concentric circles with (M_1, M_2) as the centre. When $B = 0$, we have from (52.28) that the streamlines, and therefore the magnetic lines are parallel straight lines.

We shall now show that the assumed inviscid, non-heat conducting flow with infinite electrical conductivity is homentropic as well as irrotational.

Using (52.26) in (51.14), we have

$$\frac{\partial V}{\partial \psi} = 0$$

and, therefore, equation (51.11) can be written as

$$\rho = -\frac{\dfrac{dp}{d\phi} + \dfrac{\lambda^2\mu}{V(\phi)H(\phi)}\;\dfrac{d}{d\phi}\left(\dfrac{H}{V}\right)}{V(\phi)\;\dfrac{dV}{d\phi}} = \rho(\phi)$$

Using this result and the fact that the orthogonal trajectories are isobaric curves, we get $\frac{\partial s}{\partial \psi} = 0$. This result and equation (51.13) prove that the flow is homentropic.

Finally, $\frac{\partial H}{\partial \psi} = \frac{\partial V}{\partial \psi} = 0$ prove that the flow is also irrotational. Hence, we state:

<u>Theorem 5.4.</u> If orthogonal trajectories of streamlines are isobaric curves and the orthogonal curvilinear net is isometric, then the streamlines are radial or parallel straight lines. Furthermore, the flow is homentropic and irrotational.

<u>Corollary</u>. If the velocity magnitude is constant along each orthogonal trajectory and the orthogonal curvilinear net is isometric, then the streamlines are radial or parallel straight lines.

By assumption

$$V = V(\phi)$$

Therefore, by (51.14), $\frac{\partial H}{\partial \psi} = o$ which when used in (51.12) gives us

$$p = p(\phi)$$

This condition on p along with the fact that (ϕ,ψ)net is isometric proves the required result.

Two familiar flow patterns of plane flows with straight streamlines are source flows and the straight parallel flows. Therefore, the question that arises is if these are the only plane flows with straight streamlines. We answer this question in the affirmative for flows of conducting fluids when their state equation is of the form

$$\rho = P(p)\ S(s) \tag{52.32}$$

Assuming that the streamlines ψ = constant are straight lines, we have $\alpha = \alpha(\psi)$. Using this in (52.11), we find

$$\sqrt{E} = e(\phi) \tag{52.33}$$

where $e(\phi)$ is an arbitrary function of ϕ.

Employing (52.33) in (51.12) and (51.14) respectively, we get

$$\frac{\partial p}{\partial \psi} = 0 \qquad \text{and} \qquad \frac{\partial V}{\partial \psi} = 0$$

Therefore, from (52.32) and (51.13), we have

$$\rho = A(\phi)\ B(\psi)$$

where $A(\phi) = P\big(p(\phi)\big)$ and $B(\psi) = S\big(s(\psi)\big)$.

Equation (51.10) can be written as

$$\sqrt{G} = \frac{F(\psi)}{\rho V}$$

which implies that

$$\sqrt{G} = \eta(\psi)\ \xi(\phi) \tag{52.34}$$

where $\eta(\psi) = \frac{F(\psi)}{B(\psi)}$, $\xi(\phi) = \frac{1}{A(\phi)V(\phi)}$ and $F(\psi)$ is an arbitrary function of ψ.

When (52.33) and (52.34) are used in (51.09), we obtain

$$\xi(\phi) = K_1 \int e(\phi)\ d\phi + K_2$$

and therefore

$$\sqrt{G} = \eta(\psi) \left\{ K_1 \int e(\phi)\ d\phi + K_2 \right\} \tag{52.35}$$

where K_1 and K_2 are two arbitrary constants.

Now using this expression for $\sqrt{G}$ in (52.11) and employing the fact that $\alpha = \alpha(\psi)$ only, we find

$$\alpha'(\psi) = K_1\ \eta(\psi) \tag{52.36}$$

Substituting for $\sqrt{E}$ and $\sqrt{G}$ from (52.33) and (52.35), in (52.09) we get

$$\left.\begin{aligned} &\frac{\partial z}{\partial \phi} = e(\phi)\ (\cos\alpha + i\ \sin\alpha) \\ \text{and} \\ &\frac{\partial z}{\partial \psi} = i\ \eta(\psi)\ \left[K_1 \int e(\phi)\ d\phi + K_2\right] (\cos\alpha + i\ \sin\alpha) \end{aligned}\right\} \quad (52.37)$$

Equations (52.37) together with (52.36) are satisfied by

$$z = \left[\int e(\phi)\ d\phi + \frac{K_2}{K_1}\right](\cos\alpha + i\ \sin\alpha) + D$$

when $K_1 \neq 0$ and D is an arbitrary constant. Separating into real and imaginary parts, we have

$$x - D_1 = \cos\alpha(\psi)\ \left[\int e(\phi)\ d\phi + \frac{K_2}{K_1}\right]$$

and

$$y - D_2 = \sin\alpha(\psi)\ \left[\int e(\phi)\ d\phi + \frac{K_2}{K_1}\right]$$

where $D = D_1 + i\ D_2$.

Squaring and adding

$$(x - D_1)^2 + (y - D_2)^2 = \left[\int e(\phi)\ d\phi + \frac{K_2}{K_1}\right]^2$$

which implies that the streamlines pass through the point (D_1, D_2) when $K_1 \neq 0$. When $K_1 = 0$, the equation (52.36) requires that

$$\alpha(\psi) = \text{constant}$$

i.e. the streamlines are parallel straight lines.

Hence, we have established the result:

Theorem 5.5. The only plane flows of conducting fluids with straight streamlines, having equation of state in the product form, are the simple source flows or flows in parallel straight lines.

Corollary. The only plane irrotational flows, for fluids with product equation of state, are either the source flows or flows in parallel straight lines.

The condition of irrotationality of plane flows in natural co-ordinate is

$$\frac{\partial}{\partial\psi}(\sqrt{E}\,V) = 0 \tag{52.38}$$

i.e.

$$\sqrt{E}\,V = F_1(\phi) \tag{52.39}$$

where $F_1(\phi)$ is an arbitrary function of ϕ.

Equation (51.14) can be written as

$$\frac{\sqrt{E}}{V} = F_2(\phi) \tag{52.40}$$

where $F_2(\phi)$ is an arbitrary function of ϕ.

Equations (52.39) and (52.40) give us

$$\left.\begin{aligned} &\sqrt{E} = e(\phi) \\ \text{and} \\ &V = V(\phi) \end{aligned}\right\} \tag{52.41}$$

Using (52.41) in (51.12), we find

$$p = p(\phi)$$

Therefore, for gases when equation of state is of the product form, the density function ρ is given by equations (52.32) and (51.13) as

$$\rho = P(p(\phi))\, S(s(\psi))$$

These forms of $\sqrt{E}$ and ρ are similar to those studied above and imply that the flows are either simple source flows or flows in parallel straighu line.

Section 3. On the Uniqueness of Incompressible Plane Flows.

For incompressible plane flows equation (51.01) becomes

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0$$

which implies the existence of the stream function $\psi(x,y)$ such that

$$v_1 = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v_2 = -\frac{\partial \psi}{\partial x} \tag{53.01}$$

Substitution of (51.08) into (51.06) yields

$$V^2 \frac{\partial v_1}{\partial y} - 2v_1\left\{v_1 \frac{\partial v_1}{\partial y} + v_2 \frac{\partial v_2}{\partial y}\right\} - V^2 \frac{\partial v_2}{\partial x}$$
$$+ 2v_2\left\{v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_2}{\partial x}\right\} = 0$$

In terms of ψ, the above equation becomes

$$\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] \frac{\partial^2\psi}{\partial y^2} - 2\frac{\partial\psi}{\partial y}\left\{\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y^2} + \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}\right\}$$
$$+ \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] \frac{\partial^2\psi}{\partial x^2} - 2\frac{\partial\psi}{\partial x}\left\{\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x^2}\right\} = 0 \tag{53.02}$$

Let $\vec{V}^*$, $\vec{H}^*$, p^* and ρ^* denote the flow variables of another incompressible orthogonal MFD flow having the same stream-line pattern as the original flow. For this flow the stream function ψ^* is given by

$$v_1^* = \frac{\partial\psi^*}{\partial y} \quad \text{and} \quad v_2^* = -\frac{\partial\psi^*}{\partial x} \tag{53.03}$$

where $\vec{V}^* = (v_1^*, v_2^*)$.

For the new defined dependent variables (53.02) can be written as

$$\left[\left(\frac{\partial\psi^*}{\partial x}\right)^2 + \left(\frac{\partial\psi^*}{\partial y}\right)^2\right] \frac{\partial^2\psi^*}{\partial y^2} - 2\,\frac{\partial\psi^*}{\partial y}\left\{\frac{\partial\psi^*}{\partial y}\,\frac{\partial^2\psi^*}{\partial y^2} + \frac{\partial\psi^*}{\partial x}\,\frac{\partial^2\psi^*}{\partial x\partial y}\right\}$$

$$+ \left[\left(\frac{\partial\psi^*}{\partial x}\right)^2 + \left(\frac{\partial\psi^*}{\partial y}\right)^2\right] \frac{\partial^2\psi^*}{\partial x^2} - 2\,\frac{\partial\psi^*}{\partial x}\left\{\frac{\partial\psi^*}{\partial y}\,\frac{\partial^2\psi^*}{\partial x\partial y} + \frac{\partial\psi^*}{\partial x}\,\frac{\partial^2\psi^*}{\partial x^2}\right\} = 0 \tag{53.04}$$

Necessary and sufficient conditions for both flows to have the same streamline pattern is that a functional relation, say $\psi^* = f(\psi)$ where f is assumed to be suitably differentiable, exists between the stream functions. The use of this functional relation and (53.02) in (53.04), reduce it to

$$\left(\frac{df}{d\psi}\right)^2 \frac{d^2 f}{d\psi^2} = 0$$

Integrating the above equation twice, we obtain

$$f(\psi) = A\psi + B$$

where A and B are arbitrary constants.

Hence the stream function ψ^* of any flow which has the same streamlines as that given by ψ must be of the form $\psi^* = A\psi + B$ where A and B are arbitrary constants.

This form of ψ^* implies

$$\vec{V}^* = A\,\vec{V} \quad \text{and} \quad \vec{H}^* = \frac{1}{A}\,\vec{H}$$

Now, if A_1 and A_2 are constants such that $A\,\vec{V}$, $\frac{1}{A}\,\vec{H}$,

$A_1\, p$, $A_2\, \rho$ satisfy (51.02), then

$$A^2 A_2 = A_1 = \frac{1}{A^2}$$

or $$A_1 = \frac{1}{A^2} \qquad \text{and} \qquad A_2 = \frac{1}{A^4}$$

Therefore, we have

Theorem 5.6. If $\vec{V}$, $\vec{H}$, p, ρ satisfy the equations of incompressible MFD flow with magnetic lines everywhere orthogonal to streamlines, then $A\,\vec{V}$, $\frac{1}{A}\,\vec{H}$, $\frac{1}{A^2}\,p$, $\frac{1}{A^4}\,\rho$ also give a possible flow with the same streamline pattern for all constant values of A.

Corollary. Incompressible MFD flows of the same fluid having the same streamline pattern are unique.

The assumption that the fluid is incompressible and the flows of the same fluid are considered gives us

$$\rho^* = \rho$$

or

$$A = 1$$

which implies that the flows are unique.

Section 4. Solutions of Some Particular Flows.

(A) Straight Parallel Flows. To study this problem, we choose natural co-ordinate system to be the rectangular co-ordinate system consisting of y = constant as the streamlines and x = constant as their orthogonal trajectories. For this net

$$\sqrt{E} = \sqrt{G} = 1 \qquad (54.01)$$

Using (54.01) in (51.12) and (51.14), we find

$$\frac{\partial p}{\partial y} = \frac{\partial V}{\partial y} = 0 \qquad (54.02)$$

Substitution of (54.01) in (51.11) yields

$$\rho = \frac{\frac{dp}{dx} + \frac{\lambda^2\mu}{V}\frac{d}{dx}\left(\frac{1}{V}\right)}{V(x)\frac{dV}{dx}} = \rho(x) \qquad (54.03)$$

Consideration of equations (51.13), (54.02), (54.03) and the equation of state (51.04) implies that the flow is homentropic and the equations of flow are

$$\left.\begin{aligned} &\frac{d}{dx}(\rho V) = 0 \\ &V\frac{dV}{dx} + \frac{1}{\rho}\frac{dp}{dx} - \frac{\lambda^2\mu}{\rho V^3}\frac{dV}{dx} = 0 \\ &\rho = f(p) \end{aligned}\right\} \qquad (54.04)$$

From equations (54.04), we have

$$\left.\begin{aligned} &\rho V = C_1 \\ &\rho V^2 + p + \frac{1}{2}\frac{\lambda^2\mu}{V^2} = C_2 \\ &\rho = f(p) \end{aligned}\right\} \qquad (54.05)$$

where C_1 and C_2 are arbitrary constants.

(B) <u>Vortex Flows</u>. To investigate the general vortex flows, we take polar co-ordinate system as the natural co-ordinate system.

For this choice, we have

$$\sqrt{E} = r \qquad \text{and} \qquad \sqrt{G} = 1 \tag{54.06}$$

From (51.10) and (54.06), we get

$$\frac{\partial \rho}{\partial \theta} + \frac{\rho}{V}\,\frac{\partial V}{\partial \theta} = 0 \tag{54.07}$$

Substitution of (54.06) in (51.11) yields

$$V\,\frac{\partial V}{\partial \theta} + \frac{C^2}{\rho}\,\frac{\partial \rho}{\partial \theta} - \frac{\lambda^2 \mu}{\rho V^3}\,\frac{\partial V}{\partial \theta} = 0 \tag{54.08}$$

From (54.07) and (54.08), we find

$$V = V(r) \tag{54.09}$$

whence $\quad \rho V^2 (V^2 - C^2) \neq \lambda^2 \mu$.

Since $\sqrt{E} = r$, therefore, (51.14) gives

$$V = A(\theta) r \tag{54.10}$$

where $A(\theta)$ is an arbitrary function of θ.

Equations (54.09) and (54.10) require that

$$V = K\,r \tag{54.11}$$

where K is an arbitrary constant.

From (54.07) and (54.09), we have

$$\rho = \rho(r)$$

which together with (51.04) and (51.13) implies that

$$p = p(r).$$

(C) <u>Source Flows</u>. In this problem, we again take the natural co-ordinate system to be polar co-ordinate system. For this system

$$\sqrt{E} = 1 \qquad \text{and} \qquad \sqrt{G} = r \tag{54.12}$$

Therefore, from (51.14) and (51.12), we get

$$V = V(r) \qquad \text{and} \qquad p = p(r) \tag{54.13}$$

From equations (51.11), (54.12) and (54.13), we find

$$\rho = \rho(r) \tag{54.14}$$

Integration of the flow equations yield

$$\left.\begin{aligned} r\,\rho\,V &= C_1 \\ \frac{1}{2}V^2 + G(p) + \frac{\lambda^2\mu}{\rho V^2} &= C_2 \end{aligned}\right\} \tag{54.15}$$

where $\rho = f(p)$, C_1 and C_2 are arbitrary constants and $G(p) = \int \frac{1}{f(p)}\,dp$.

CHAPTER VI

VISCOUS MHD NON-ALIGNED FLOWS

In this chapter, we extend M.H. Martin's (1971) theorems for non-conducting, incompressible, viscous flows to non-aligned, incompressible, viscous MHD flows.

Section 1. Flow Equations.

For steady, thermally non-conducting, incompressible, viscous (η = constant) and electrically infinitely conducting flows, the fundamental equations of MFD reduce to

$$\operatorname{div} \vec{V} = 0$$

$$\rho(\vec{V} \cdot \operatorname{grad})\vec{V} + \operatorname{grad} p = \eta \nabla^2 \vec{V} + \mu(\operatorname{curl} \vec{H}) \times \vec{H}$$

$$\operatorname{curl} (\vec{V} \times \vec{H}) = \vec{0}$$

$$\operatorname{div} \vec{H} = 0$$

where ∇^2 is the Laplacian operator. In case of two-dimensional flows, when $\vec{H}$ is in the plane of flow, these equations become

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{61.01}$$

$$\rho\left(v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y}\right) + \frac{\partial p}{\partial x} = \eta\left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}\right) - \mu H_2\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right) \tag{61.02}$$

$$\rho\left(v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y}\right) + \frac{\partial p}{\partial y} = \eta\left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2}\right) + \mu H_1\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right) \tag{61.03}$$

$$v_1 H_2 - v_2 H_1 = K \tag{61.04}$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \tag{61.05}$$

where $\vec{V} = (v_1, v_2)$, $\vec{H} = (H_1, H_2)$ and K is an arbitrary constant.

Throughout this chapter, we assume that the streamlines are nowhere parallel to the magnetic lines of force. Therefore, we have

$$K \neq 0$$

Equations (61.01) to (61.05) is a system of non-linear partial differential equations of highest order two in five dependent variables i.e. v_1, v_2, H_1, H_2 and p.

On introducing the functions

$$\left.\begin{aligned} \omega &= \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \\ \Omega &= \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \\ h &= \frac{\rho}{2} V^2 + p \\ V^2 &= v_1^2 + v_2^2 \end{aligned}\right\} \tag{61.06}$$

the equation (61.02) can be written as

$$\rho\left(v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y}\right) + \frac{\partial h}{\partial x} - \rho\left(v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_2}{\partial x}\right)$$
$$= \eta\left[\frac{\partial^2 v_1}{\partial x^2} + \left(\frac{\partial^2 v_2}{\partial x \partial y} - \frac{\partial \omega}{\partial y}\right)\right] - \mu\, \Omega\, H_2$$

or

$$- \rho\, v_2 \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) + \frac{\partial h}{\partial x} = - \eta \frac{\partial \omega}{\partial y} + \eta \frac{\partial}{\partial x} \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \right) - \mu\, \Omega\, H_2$$

Using (61.01), (61.06), we get

$$\eta \frac{\partial \omega}{\partial y} - \rho\, \omega\, v_2 + \mu\, \Omega\, H_2 = - \frac{\partial h}{\partial x}$$

Similarly, (61.03) gives us

$$\eta \frac{\partial \omega}{\partial x} - \rho\, \omega\, v_1 + \mu\, \Omega\, H_1 = \frac{\partial h}{\partial y}$$

Therefore, the five partial differential equations (61.01) to (61.05) can be replaced by the following seven partial differential equations

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \qquad (61.07)$$

$$\eta \frac{\partial \omega}{\partial y} - \rho\, \omega\, v_2 + \mu\, \Omega\, H_2 = - \frac{\partial h}{\partial x} \qquad (61.08)$$

$$\eta \frac{\partial \omega}{\partial x} - \rho\, \omega\, v_1 + \mu\, \Omega\, H_1 = \frac{\partial h}{\partial y} \qquad (61.09)$$

$$v_1\, H_2 - v_2\, H_1 = K \qquad (61.10)$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \qquad (61.11)$$

$$\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = \omega \qquad (61.12)$$

$$\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = \Omega \qquad (61.13)$$

The set of equations (61.07) to (61.13) is a system of non-linear partial differential equations in seven dependent variables i.e. v_1, v_2, H_1, H_2, ω, Ω and h. Although the number of equations and dependent variables has increased by two, the order has decreased from two to one.

Equations (61.07) and (61.11) respectively imply the existence of the stream function $\psi(x,y)$ and the magnetic function $\phi(x,y)$ such that

$$v_2 = -\frac{\partial \psi}{\partial x} \quad , \quad v_1 = \frac{\partial \psi}{\partial y} \qquad (61.14)$$

and

$$H_2 = \frac{\partial \phi}{\partial x} \quad , \quad H_1 = -\frac{\partial \phi}{\partial y} \qquad (61.15)$$

We now assume that the curves ψ = constant and the curves ϕ = constant form the curvilinear co-ordinate system discussed in section 2 of chapter II.

Using (61.14) and (61.15) in (61.10), we find

$$\frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial y} = \frac{\partial(\phi,\psi)}{\partial(x,y)} = \frac{1}{J} = K \neq 0 \qquad (61.16)$$

where J is defined by (22.05).

Equation (61.16) implies that if we are given

$x = x(\phi,\psi)$; $v_1 = v_1(\phi,\psi)$; $H_1 = H_1(\phi,\psi)$; $\omega = \omega(\phi,\psi)$;

$y = y(\phi,\psi)$; $v_2 = v_2(\phi,\psi)$; $H_2 = H_2(\phi,\psi)$; $\Omega = \Omega(\phi,\psi)$;

$h = h(\phi,\psi)$

we can find

$$v_1 = v_1(x,y);\ H_1 = H_1(x,y);\ \omega = \omega(x,y)$$

$$v_2 = v_2(x,y);\ H_2 = H_2(x,y);\ \Omega = \Omega(x,y);\ h = h(x,y).$$

From equations (22.06) and (61.16), we have

$$E\ G - F^2 = \frac{1}{K^2} \tag{61.17}$$

Section 2. New Form for the Fundamental Equations.

In this section, we shall obtain such a form of the flow equations that their solution gives us v_1, v_2, H_1, H_2, ω, Ω and h as functions of ϕ and ψ.

Solenoidal Condition on $\vec{H}$. Using (22.04) in (61.15), we get

$$\frac{\partial x}{\partial \psi} = J\, H_1 \quad , \quad \frac{\partial y}{\partial \psi} = J\, H_2 \qquad (62.01)$$

Let θ be the angle made by the magnetic field $\vec{H}$ with x-axis. Then the components H_1 and H_2 of $\vec{H}$ can be written as

$$H_1 = H \cos\theta \quad , \quad H_2 = H \sin\theta \qquad (62.02)$$

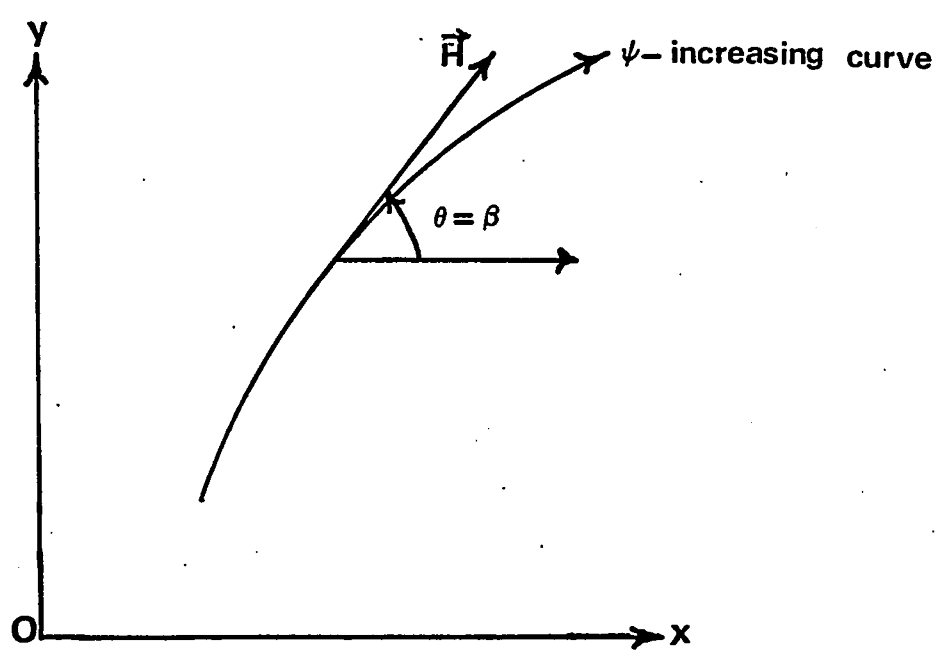
where $H = |\vec{H}|$.

Therefore, (62.01) can be written as

$$\frac{\partial x}{\partial \psi} = J\, H \cos\theta \quad , \quad \frac{\partial y}{\partial \psi} = J\, H \sin\theta \qquad (62.03)$$

Now two cases arise

1st Case. When $\theta = \beta$.



β is the angle made by the tangent to the curve ϕ = constant, directed in the sense of increasing ψ, with x-axis. In this case (62.03) becomes

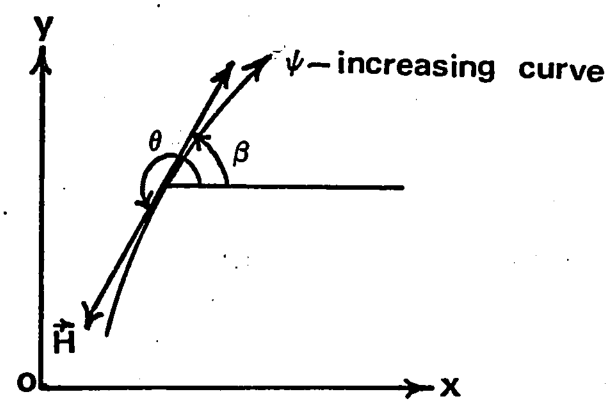
$$\frac{\partial x}{\partial \psi} = J\ H \cos\beta, \quad \frac{\partial y}{\partial \psi} = J\ H \sin\beta \tag{62.04}$$

From (22.07) and (62.04), we get

$$J\ H = \sqrt{G} \tag{62.05}$$

i.e. $J > 0$

<u>2nd Case</u>. <u>When $\theta = \beta + \pi$</u>.



From (62.03), we have

$$\frac{\partial x}{\partial \psi} = -\ J\ H \cos\beta, \quad \frac{\partial y}{\partial \psi} = -\ J\ H \sin\beta \tag{62.06}$$

Equations (62.06) together with (22.07) give

$$-\ J\ H = \sqrt{G} \tag{62.07}$$

$$\text{i.e. } J < 0$$

From the above two cases, we conclude that the magnetic field acts along the magnetic lines towards higher or lower parameter values ψ accordingly as J is positive or negative. In either case (22.06) requires that

$$W\ H = \sqrt{G} \tag{62.08}$$

Equations (62.01) and (22.07) imply that

$$H_1 + i\, H_2 = \frac{\sqrt{G}}{J}\, e^{i\beta} \tag{62.09}$$

Equation of Continuity. Using (22.04) in (61.14), we get

$$\frac{\partial x}{\partial \phi} = J\, v_1 \quad , \quad \frac{\partial y}{\partial \phi} = J\, v_2$$

Proceeding exactly as in the case of 'Solenoidal Condition on $\vec{H}$' we find that the fluid flows along the streamlines towards higher or lower parameter values ϕ accordingly as J is positive or negative and

$$W\, V = \sqrt{E} \tag{62.10}$$

$$v_1 + i\, v_2 = \frac{\sqrt{E}}{J}\, e^{i\alpha} \tag{62.11}$$

where α is the angle between the tangent to the co-ordinate line ψ = constant, directed in the sense of increasing ϕ, with x-axis.

Eliminating V between (62.10) and (61.06), we obtain

$$h = \frac{\rho}{2}\, \frac{E}{W^2} + p$$

Therefore, if we know E, F, G and h, we can find p.

The Function Ω. By definition

$$\Omega = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}$$

$$= \left(\frac{\partial H_2}{\partial \phi}\, \frac{\partial \phi}{\partial x} + \frac{\partial H_2}{\partial \psi}\, \frac{\partial \psi}{\partial x}\right) - \left(\frac{\partial H_1}{\partial \phi}\, \frac{\partial \phi}{\partial y} + \frac{\partial H_1}{\partial \psi}\, \frac{\partial \psi}{\partial y}\right)$$

Using (22.04), we have

$$J\Omega = \left(\frac{\partial H_2}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial H_2}{\partial \psi}\frac{\partial y}{\partial \phi}\right) + \left(\frac{\partial H_1}{\partial \phi}\frac{\partial x}{\partial \psi} - \frac{\partial H_1}{\partial \psi}\frac{\partial x}{\partial \phi}\right)$$

On substituting

$$H_1 = \pm H \cos\beta \quad , \quad H_2 = \pm H \sin\beta$$

we find

$$\pm J\Omega = \left[\left(\frac{\partial H}{\partial \phi}\sin\beta + H\cos\beta\frac{\partial \beta}{\partial \phi}\right)\frac{\partial y}{\partial \psi} - \left(\frac{\partial H}{\partial \psi}\sin\beta + H\cos\beta\frac{\partial \beta}{\partial \psi}\right)\frac{\partial y}{\partial \phi}\right] + \left[\left(\frac{\partial H}{\partial \phi}\cos\beta - H\sin\beta\frac{\partial \beta}{\partial \phi}\right)\frac{\partial x}{\partial \psi} - \left\{\frac{\partial H}{\partial \psi}\cos\beta - H\sin\beta\frac{\partial \beta}{\partial \psi}\right\}\frac{\partial x}{\partial \phi}\right]$$

Using (22.07), (22.09) and (22.10), we get

$$\pm J\Omega = \frac{1}{\sqrt{G}}\left[G\frac{\partial H}{\partial \phi} - F\frac{\partial H}{\partial \psi} + H J\frac{\partial \beta}{\partial \psi}\right]$$

or

$$\sqrt{G}\, W\,\Omega = G\frac{\partial H}{\partial \phi} - F\frac{\partial H}{\partial \psi} + H J\frac{\partial \beta}{\partial \psi} \tag{62.12}$$

Eliminating H between (62.08) and (62.12), we have

$$\sqrt{G}\, W\,\Omega = \frac{GW}{\sqrt{G}}\frac{\partial}{\partial \phi}\left(\frac{G}{2W^2}\right) - \frac{FW}{\sqrt{G}}\frac{\partial}{\partial \psi}\left(\frac{G}{2W^2}\right) + \frac{\sqrt{G}}{W} J\frac{\partial \beta}{\partial \psi} \tag{62.13}$$

Equation (62.13), on using (22.17) and (22.21), gives

$$\sqrt{G}\, W\,\Omega = \frac{1}{W^2}\frac{GW}{\sqrt{G}}(G\gamma^2_{22} - F\gamma^2_{12}) - \frac{1}{W^2}\frac{FW}{\sqrt{G}}(G\gamma^2_{12} - F\gamma^2_{11}) + \frac{W}{G}\sqrt{G}\,\gamma^2_{11}$$

or

$$W\,\Omega = \frac{1}{W}\left\{G\gamma^2_{22} - 2F\gamma^2_{12} + \frac{F^2}{G}\gamma^2_{11}\right\} + \frac{1}{W}\left\{E - \frac{F^2}{G}\right\}\gamma^2_{11}$$

$$= \frac{1}{W}\left[G\gamma^2_{22} - 2F\gamma^2_{12} + E\gamma^2_{11}\right]$$

From (22.22), we get

$$\Omega = \frac{1}{W}\left[\frac{\partial}{\partial\phi}\left(\frac{G}{W}\right) - \frac{\partial}{\partial\psi}\left(\frac{F}{W}\right)\right] \tag{62.14}$$

<u>The Vorticity ω.</u> On changing the independent variables from x,y to ϕ,ψ, we find

$$J\,\omega = \left(\frac{\partial v_2}{\partial\phi}\frac{\partial y}{\partial\psi} - \frac{\partial v_2}{\partial\psi}\frac{\partial y}{\partial\phi}\right) + \left(\frac{\partial v_1}{\partial\phi}\frac{\partial x}{\partial\psi} - \frac{\partial v_1}{\partial\psi}\frac{\partial x}{\partial\phi}\right)$$

Proceeding as above and using (22.24), (22.25), we obtain

$$\omega = \frac{1}{W}\left[\frac{\partial}{\partial\phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial\psi}\left(\frac{E}{W}\right)\right] \tag{62.15}$$

<u>Equations of momentum</u>. Equation (61.08) can be written as

$$\eta\left\{\frac{\partial\omega}{\partial\phi}\frac{\partial\phi}{\partial y} + \frac{\partial\omega}{\partial\psi}\frac{\partial\psi}{\partial y}\right\} + \rho\,\omega\,\frac{\partial\psi}{\partial x} + \mu\,\Omega\,\frac{\partial\phi}{\partial x}$$

$$= -\left\{\frac{\partial h}{\partial\phi}\frac{\partial\phi}{\partial x} + \frac{\partial h}{\partial\psi}\frac{\partial\psi}{\partial x}\right\} \tag{62.16}$$

where (61.14) and (61.15) have been used to eliminate v_2 and H_2.

On using (22.04), (62.16) becomes

$$\eta\left\{-\frac{\partial\omega}{\partial\phi}\frac{\partial x}{\partial\psi} + \frac{\partial\omega}{\partial\psi}\frac{\partial x}{\partial\phi}\right\} - \rho\,\omega\,\frac{\partial y}{\partial\phi} + \mu\,\Omega\,\frac{\partial y}{\partial\psi}$$

$$= \left\{-\frac{\partial h}{\partial\phi}\frac{\partial y}{\partial\psi} + \frac{\partial h}{\partial\psi}\frac{\partial y}{\partial\phi}\right\} \tag{62.17}$$

Similarly, (61.09) gives us

$$\eta\left\{\frac{\partial\omega}{\partial\phi}\frac{\partial y}{\partial\psi} - \frac{\partial\omega}{\partial\psi}\frac{\partial y}{\partial\phi}\right\} - \rho\,\omega\,\frac{\partial x}{\partial\phi} + \mu\,\Omega\,\frac{\partial x}{\partial\psi}$$

$$= \left\{-\frac{\partial h}{\partial\phi}\frac{\partial x}{\partial\psi} + \frac{\partial h}{\partial\psi}\frac{\partial x}{\partial\phi}\right\} \tag{62.18}$$

Multiplying (62.17) by $\frac{\partial y}{\partial \phi}$, (62.18) by $\frac{\partial x}{\partial \phi}$ and adding,
we get

$$\eta\ J \frac{\partial \omega}{\partial \phi} - \rho\ \omega\ E + \mu\ \Omega\ F = -\ F \frac{\partial h}{\partial \phi} + E \frac{\partial h}{\partial \psi}$$

or

$$-\ F \left\{ \frac{\partial h}{\partial \phi} + \mu\ \Omega \right\} + E \left\{ \frac{\partial h}{\partial \psi} + \rho\ \omega \right\} = \eta\ J \frac{\partial \omega}{\partial \phi} \qquad (62.19)$$

where E, F and J are given by (22.03) and (22.05).
Again, multiplying (62.17) by $\frac{\partial y}{\partial \psi}$, (62.18) by $\frac{\partial x}{\partial \psi}$ and adding,
we obtain

$$\eta\ J \frac{\partial \omega}{\partial \psi} - \rho\ \omega\ F + \mu\ \Omega\ G = -\ G \frac{\partial h}{\partial \phi} + F \frac{\partial h}{\partial \psi}$$

or

$$G \left\{ \frac{\partial h}{\partial \phi} + \mu\ \Omega \right\} - F \left\{ \frac{\partial h}{\partial \psi} + \rho\ \omega \right\} = -\ \eta\ J \frac{\partial \omega}{\partial \psi} \qquad (62.20)$$

Equations (62.19) and (62.20) are the new forms for the momentum equations.

Equations (62.19) and (62.20) can be written in another form by eliminating $\frac{\partial h}{\partial \psi}$ and $\frac{\partial h}{\partial \phi}$ respectively between them and the resulting equations are

$$\left.\begin{aligned} &\frac{\partial h}{\partial \phi} + \mu\ \Omega = \frac{\eta}{J} \left\{ F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right\} \\ \text{and}\quad &\\ &\frac{\partial h}{\partial \psi} + \rho\ \omega = \frac{\eta}{J} \left\{ G \frac{\partial \omega}{\partial \phi} - F \frac{\partial \omega}{\partial \psi} \right\} \end{aligned}\right\} \qquad (62.21)$$

Summing up the results of this section, we have

<u>Theorem 6.1</u>. When the streamlines ψ = constant and the magnetic lines ϕ = constant of steady, plane flow of a viscous,

infinitely conducting (electrically), incompresssible fluid are taken as the curvilinear co-ordinate system ϕ,ψ in the physical plane, the set of seven partial differential equations (61.07) to (61.13) for v_1, v_2, H_1, H_2, ω, Ω and h as functions of x,y may be replaced by the system

$$\left.\begin{aligned}
&- F \left\{\frac{\partial h}{\partial \phi} + \mu\ \Omega\right\} + E\left\{\frac{\partial h}{\partial \psi} + \rho\ \omega\right\} = \eta\ J\ \frac{\partial \omega}{\partial \phi} \\
&G \left\{\frac{\partial h}{\partial \phi} + \mu\ \Omega\right\} - F\left\{\frac{\partial h}{\partial \psi} + \rho\ \omega\right\} = -\ \eta\ J\ \frac{\partial \omega}{\partial \psi} \\
&\frac{\partial}{\partial \psi}\left\{\frac{J}{G}\ \gamma^2_{12}\right\} - \frac{\partial}{\partial \phi}\left\{\frac{J}{G}\ \gamma^2_{11}\right\} = 0 \\
&\Omega = \frac{1}{W}\left\{\frac{\partial}{\partial \phi}\left(\frac{G}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{F}{W}\right)\right\} \\
&\omega = \frac{1}{W}\left\{\frac{\partial}{\partial \phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{W}\right)\right\} \\
&W^2 = J^2 = EG - F^2 = \frac{1}{K^2}
\end{aligned}\right\} \qquad (62.22)$$

of six partial differential equations for E, F, G, ω, Ω and h as functions of ϕ,ψ. Here E, F, G are given by

$$ds^2 = E\ d\phi^2 + 2F\ d\phi\ d\psi + G\ d\psi^2$$

where ds is the element of arc length in the physical plane. The Jacobian, J is positive or negative as the parameter ψ increases or decreases in the direction of the magnetic field vector $\vec{H}$.

Given a solution

$$E = E(\phi,\psi); \quad F = F(\phi,\psi); \quad G = G(\phi,\psi)$$

$$\omega = \omega(\phi,\psi); \quad \Omega = \Omega(\phi,\psi); \quad h = h(\phi,\psi)$$

of the system (62.22), we can find x,y as functions of ϕ,ψ from

$$z = x + iy = \int \frac{e^{i\beta}}{\sqrt{G}} \left\{ (F - i\, J)d\phi + G\, d\psi \right\}$$

where

$$\beta = \int \frac{J}{G} \left(\gamma^{2}_{12}\; d\phi + \gamma^{2}_{11}\; d\psi\right)$$

and thus obtain E, F, G, ω, Ω and h as functions of x,y because

$$0 < |J| < \infty$$

Once we obtain E, F, G and h as functions of x,y then H_1, H_2, v_1, v_2 and p as functions of x,y are given by

$$H_1 + i\, H_2 = \frac{\sqrt{G}}{J}\, e^{i\beta}$$

$$v_1 + i\, v_2 = \frac{\sqrt{E}}{J}\, e^{i\alpha}$$

and

$$p = h - \frac{\rho}{2}\; \frac{E}{W^2}$$

Section 3. Another Form for Fundamental Equations.

Equations (62.21) can be rewritten as

$$\frac{\partial h}{\partial \phi} = -\mu\,\Omega + \frac{\eta}{J}\left\{F\,\frac{\partial \omega}{\partial \phi} - E\,\frac{\partial \omega}{\partial \psi}\right\} \tag{63.01}$$

and

$$\frac{\partial h}{\partial \psi} = -\rho\,\omega + \frac{\eta}{J}\left\{G\,\frac{\partial \omega}{\partial \phi} - F\,\frac{\partial \omega}{\partial \psi}\right\} \tag{63.02}$$

Differentiating (63.01) partially with respect to ψ, (63.02) partially with respect to ϕ and using the condition that the second order mixed derivatives of h with respect to ϕ and ψ are independent of the order of differentiation, we find

$$\eta\,J\,\Delta_2\omega + \mu\,\frac{\partial \Omega}{\partial \psi} - \rho\,\frac{\partial \omega}{\partial \phi} = 0 \tag{63.03}$$

or

$$\eta\,W\,\Delta_2\omega \pm \mu\,\frac{\partial \Omega}{\partial \psi} \mp \rho\,\frac{\partial \omega}{\partial \phi} = 0$$

where

$$\Delta_2\omega \equiv \frac{1}{J}\left[\frac{\partial}{\partial \phi}\left\{\frac{1}{J}\left(G\,\frac{\partial \omega}{\partial \phi} - F\,\frac{\partial \omega}{\partial \psi}\right)\right\} + \frac{\partial}{\partial \psi}\left\{\frac{1}{J}\left(E\,\frac{\partial \omega}{\partial \psi} - F\,\frac{\partial \omega}{\partial \phi}\right)\right\}\right] \tag{63.04}$$

Therefore, we have reduced the system of equations (62.22) to five equations

$$\eta\,J\,\Delta_2\omega + \mu\,\frac{\partial \Omega}{\partial \psi} - \rho\,\frac{\partial \omega}{\partial \phi} = 0$$

$$\Omega = \frac{1}{J}\left\{\frac{\partial}{\partial \phi}\left(\frac{G}{J}\right) - \frac{\partial}{\partial \psi}\left(\frac{F}{J}\right)\right\} \tag{63.05}$$

$$\omega = \frac{1}{J}\left\{\frac{\partial}{\partial \phi}\left(\frac{F}{J}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{J}\right)\right\} \tag{63.06}$$

$$EG - F^2 = \frac{1}{K^2} \tag{63.07}$$

and

$$\frac{\partial}{\partial\psi}\left[\frac{J}{G}\gamma^2_{12}\right] - \frac{\partial}{\partial\phi}\left[\frac{J}{G}\gamma^2_{11}\right] = 0 \qquad (63.08)$$

in five dependent variables E, F, G, ω and Ω. If the solution to these equations is given, we can find $h = h(\phi,\psi)$ from the equations of momentum.

We shall now study two examples in which the curves ψ = constant and the curves ϕ = constant form an orthogonal curvilinear co-ordinate system.

1st Example. In this example, we prescribe the streamlines to be straight lines. Let us assume that they are not parallel but envelope to a curve Γ. We shall now take the tangent lines to the curve Γ and their orthogonal trajectories, the involutes of Γ as the system of orthogonal curvilinear co-ordinates. The square of the element of arc length ds in this orthogonal curvilinear co-ordinate system is given by

$$ds^2 = ds_1^2 + ds_2^2$$

where ds_1 is the element of arc length of the involute and ds_2 is the element of arc length of the tangent.

The element of arc length of the involute is

$$ds_1 = (\xi - \sigma)\ \kappa\ d\sigma$$

where σ denotes the arc length, κ the curvature of the curve Γ and ξ is a parameter constant on each involute. Therefore, we have

$$ds^2 = d\xi^2 + (\xi - \sigma)^2\ \kappa^2\ d\sigma^2$$

But

$$\kappa = \frac{d\eta}{d\sigma}$$

where η is the angle subtended by the tangent line with x-axis. Hence, we have

$$ds^2 = d\xi^2 + (\xi - \sigma)^2 \, d\eta^2 \tag{63.09}$$

where $\sigma = \sigma(\eta)$.

In this co-ordinate system, the co-ordinate curves ξ = constant are the involutes of the curve Γ and the curves η = constant its tangent lines.

We now investigate the flows for which

$$\phi = \phi(\xi) \qquad , \qquad \psi = \psi(\eta) \tag{63.10}$$

Using (63.10) in (22.02), we get

$$ds^2 = (\phi')^2 \, E \, d\xi^2 + 2F \, \phi' \, \psi' \, d\xi \, d\eta + G(\psi')^2 \, d\eta^2 \tag{63.11}$$

Comparing (63.09) and (63.11), we find

$$\left.\begin{aligned} &E = \left(\frac{1}{\phi'}\right)^2 \quad , \quad F = 0, \quad G = \left\{\frac{\xi - \sigma(\eta)}{\psi'}\right\}^2 \\ &\text{and} \qquad\qquad J = \frac{\xi - \sigma(\eta)}{\phi' \, \psi'} \end{aligned}\right\} \tag{63.12}$$

Since $F = 0$, therefore (22.12) give

$$\gamma^2_{11} = \frac{1}{2W^2} \, G \, \frac{\partial G}{\partial \phi}$$

and

$$\gamma^2_{12} = - \frac{1}{2W^2} \, G \, \frac{\partial E}{\partial \psi}$$

Using (63.12), we get

$$\left.\begin{aligned} \gamma^2_{11} &= \frac{\xi - \sigma(\eta)}{\psi'}\,\phi' \\ \text{and} \qquad & \\ \gamma^2_{12} &= 0 \end{aligned}\right\} \qquad (63.13)$$

Substituting for G, J, γ^2_{11} and γ^2_{12} from (63.12) and (63.13) in (63.08), we find that it is automatically satisfied.

From (63.05), we have

$$\Omega = \frac{1}{J}\frac{\partial}{\partial\phi}\left(\frac{G}{J}\right) = \frac{\phi'\psi'}{\xi-\sigma(\eta)}\,\frac{\partial}{\partial\phi}\left[\frac{(\xi - \sigma)\phi'}{\psi'}\right]$$

where (63.12) is used.

Therefore, the function Ω is given by

$$\Omega = \frac{1}{\xi - \sigma(\eta)}\,\frac{\partial}{\partial\xi}\left\{\phi'\left(\xi - \sigma(\eta)\right)\right\} \qquad (63.14)$$

Similarly, (63.06) gives us

$$\omega = -\frac{1}{J}\frac{\partial}{\partial\psi}\left(\frac{E}{J}\right) = -\frac{1}{\xi - \sigma}\,\frac{\partial}{\partial\eta}\left[\frac{\psi'}{\xi - \sigma}\right] \qquad (63.15)$$

When F = 0, (63.04) can be written as

$$\begin{aligned} \Delta_2\omega &= \frac{1}{J}\left[\frac{\partial}{\partial\phi}\left\{\frac{1}{J}\,G\,\frac{\partial\omega}{\partial\phi}\right\} + \frac{\partial}{\partial\psi}\left\{\frac{1}{J}\,G\,\frac{\partial\omega}{\partial\psi}\right\}\right] \\ &= \frac{\phi'\psi'}{\xi - \sigma}\left[\frac{1}{\phi'}\,\frac{\partial}{\partial\xi}\left\{\frac{\xi - \sigma}{\psi'}\,\frac{\partial\omega}{\partial\xi}\right\} + \frac{1}{\psi'}\,\frac{\partial}{\partial\eta}\left\{\frac{1}{\phi'(\xi - \sigma)}\,\frac{\partial\omega}{\partial\eta}\right\}\right] \\ &= \frac{1}{\xi - \sigma}\left[\frac{\partial}{\partial\xi}\left\{(\xi - \sigma)\,\frac{\partial\omega}{\partial\xi}\right\} + \frac{\partial}{\partial\eta}\left\{\frac{1}{\xi - \sigma}\,\frac{\partial\omega}{\partial\eta}\right\}\right] \end{aligned} \qquad (63.16)$$

Using (63.16), (63.03) becomes

$$\eta\left[\frac{\partial}{\partial\xi}\left\{(\xi-\sigma)\frac{\partial\omega}{\partial\xi}\right\}+\frac{\partial}{\partial\eta}\left\{\frac{1}{\xi-\sigma}\frac{\partial\omega}{\partial\eta}\right\}\right]+\phi'\mu\frac{\partial\Omega}{\partial\eta}$$
$$-\rho\,\psi'\frac{\partial\omega}{\partial\xi}=0 \qquad (63.17)$$

Eliminating Ω and ω between (63.14), (63.15) and (63.17),

we get

$$\eta\left[\frac{\partial}{\partial\xi}\left\{(\xi-\sigma)\left(\frac{3\psi'\sigma'}{(\xi-\sigma)^4}+\frac{2\psi''}{(\xi-\sigma)^3}\right)\right\}\right.$$
$$+\frac{\partial}{\partial\eta}\left\{\frac{1}{\xi-\sigma}\left(-\frac{\psi'\sigma''}{(\xi-\sigma)^3}-\frac{3\psi'(\sigma')^2}{(\xi-\sigma)^4}-\frac{\psi'''}{(\xi-\sigma)^2}\right.\right.$$
$$\left.\left.\left.-\frac{3\psi''\sigma'}{(\xi-\sigma)^3}\right)\right\}\right]+\phi'\mu\frac{\phi'\sigma'}{(\xi-\sigma)^2}-\rho\,\psi'\left\{\frac{3\psi'}{(\xi-\sigma)^4}+\frac{2\psi''}{(\xi-\sigma)^3}\right\}$$
$$=0$$

or

$$15\eta\,\psi'(\sigma')^3+(\xi-\sigma)\,[10\eta\,\psi'\,\sigma'\,\sigma''+15\eta\,\psi''(\sigma')^2]$$
$$+(\xi-\sigma)^2\,[9\eta\,\psi'\,\sigma'+4\eta\,\psi''\,\sigma''+\eta\,\psi'\,\sigma'''+6\eta\,\psi'''\,\sigma'$$
$$+3\rho(\psi')^2\,\sigma']+(\xi-\sigma)^3\,[4\eta\,\psi''+\eta\,\psi^{(iv)}+2\rho\,\psi'\,\psi''$$
$$-(\xi-\sigma)^4\,[\mu(\phi')^2\,\sigma']=0 \qquad (63.18)$$

The curve Γ appears as the curve $\xi=\sigma(\eta)$ in the plane of variables ξ,η. For the relation (63.18) to hold identically, it must hold on the curve $\xi=\sigma(\eta)$ and therefore, we have

$$\sigma'=0$$
$$\kappa\to\infty$$

which implies

<u>Theorem 6.2</u>. If the streamlines in two dimensional MFD flow of a viscous fluid are straight lines but not parallel, then they must be concurrent.

<u>2nd Example</u>. In this example, we consider the involutes to the curve Γ as the streamlines and the tangents to the curve Γ as the magnetic lines.

As in the previous example, square of the element of arc length in this orthogonal curvilinear co-ordinate system is

$$ds^2 = d\xi^2 + (\xi - \sigma)^2 \, d\eta^2 \tag{63.19}$$

For the flows under investigation

$$\phi = \phi(\eta) \qquad \text{and} \qquad \psi = \psi(\xi) \tag{63.20}$$

Using (63.20) in (22.02), we get

$$ds^2 = (\phi')^2 \, E \, d\eta^2 + 2F \, \phi' \psi' \, d\xi \, d\eta + G(\psi')^2 \, d\xi^2 \tag{63.21}$$

Comparing (63.21) with (63.19), we obtain

$$\left.\begin{aligned} E &= \left(\frac{\xi - \sigma}{\phi'}\right)^2, \quad F = 0, \quad G = \left(\frac{1}{\psi'}\right)^2 \\ \text{and} \qquad J &= \frac{\xi - \sigma}{\phi' \psi'} \end{aligned}\right\} \tag{63.22}$$

Equation (63.08) is again automatically satisfied and (63.05) gives

$$\Omega = \frac{1}{J} \frac{\partial}{\partial \phi} \left(\frac{G}{J}\right)$$

or

$$\Omega = \frac{\phi' \psi'}{\xi - \sigma} \frac{\partial}{\partial \phi}\left(\frac{\phi'}{\psi' (\xi - \sigma)} \right)$$

where (63.22) is used.

Using (63.20), we find

$$\Omega = \frac{1}{\xi - \sigma} \frac{\partial}{\partial \eta}\left(\frac{\phi'}{\xi - \sigma} \right) \tag{63.23}$$

Similarly, from (63.06), we get

$$\omega = - \frac{1}{J} \frac{\partial}{\partial \psi}\left(\frac{E}{J} \right)$$

$$= - \frac{1}{\xi - \sigma} \frac{\partial}{\partial \xi} [\psi' (\xi - \sigma)] \tag{63.24}$$

where (63.20) and (63.22) are used.

Since, the curvilinear co-ordinates are orthogonal, therefore (63.04) becomes

$$\Delta_2 \omega = \frac{1}{J}\left[\frac{\partial}{\partial \phi}\left\{ JG \frac{\partial \omega}{\partial \phi} \right\} + \frac{\partial}{\partial \psi}\left\{ \frac{1}{J} G \frac{\partial \omega}{\partial \psi} \right\} \right]$$

$$= \frac{\phi' \psi'}{\xi - \sigma}\left[\frac{1}{\phi'} \frac{\partial}{\partial \eta}\left\{ \frac{1}{\psi' (\xi - \sigma)} \frac{\partial \omega}{\partial \eta} \right\} + \frac{1}{\psi'} \frac{\partial}{\partial \xi}\left\{ \frac{\xi - \sigma}{\phi'} \frac{\partial \omega}{\partial \xi} \right\} \right]$$

$$= \frac{1}{\xi - \sigma}\left[\frac{\partial}{\partial \eta}\left\{ \frac{1}{\xi - \sigma} \frac{\partial \omega}{\partial \eta} \right\} + \frac{\partial}{\partial \xi}\left\{ (\xi - \sigma) \frac{\partial \omega}{\partial \xi} \right\} \right] \tag{63.25}$$

Equation (63.03), on using (63.25), becomes

$$\eta \frac{\partial}{\partial \xi}\left[(\xi - \sigma) \frac{\partial \omega}{\partial \xi} \right] + \eta \frac{\partial}{\partial \eta}\left[\frac{1}{\xi - \sigma} \frac{\partial \omega}{\partial \eta} \right] + \mu \phi' \frac{\partial \Omega}{\partial \xi}$$

$$- \rho \psi' \frac{\partial \omega}{\partial \eta} = 0 \tag{63.26}$$

Elimination of Ω and ω between (63.23), (63.24) and

(63.26) gives

$$\eta \frac{\partial}{\partial \xi}\left[(\xi - \sigma)\left\{-\psi''' - \frac{\psi''}{\xi - \sigma} + \frac{\psi'}{(\xi - \sigma)^2}\right\}\right] + \eta\left[\frac{1}{\xi - \sigma}\left\{-\frac{\psi'\sigma'}{(\xi - \sigma)^2}\right\}\right]$$

$$+ \mu\,\phi'\left\{-\frac{2\phi''}{(\xi - \sigma)^3} - \frac{3\phi'\sigma'}{(\xi - \sigma)^4}\right\} - \rho\,\psi'\left\{-\frac{\psi'\sigma'}{(\xi - \sigma)^2}\right\} = 0$$

or

$$3\sigma'[\eta\,\psi'\,\sigma' + \mu(\phi')^2] + (\xi - \sigma)\;[\eta\,\psi'\,\sigma'' + 2\mu\,\phi'\,\phi'']$$

$$+ (\xi - \sigma)^2\;[\eta\,\psi' - (\psi')^2\,\sigma'] - \psi''\,(\xi - \sigma)^3 + 2\psi'''(\xi - \sigma)^4$$

$$+ \psi^{(iv)}\,(\xi - \sigma)^5 \equiv 0 \qquad (63.27)$$

Using the same argument as in the 1st example, we have either $\sigma' = 0$ i.e. $\kappa \to \infty$ or

$$\psi' = -\mu\frac{(\phi')^2}{\eta\,\sigma'} = A, \text{ a constant.}$$

If ψ' = constant, the equation (63.22) implies that G = constant and thus from (63.07) we have E = constant. This is not possible. Therefore, we have

Theorem 6.3. If the streamlines in plane MFD flow of a viscous fluid are involutes of a curve Γ, then the streamlines are concentric circles.

Section 4. Radial and Vortex Flows.

In this section we study the radial and vortex flows when the magnetic field vector $\vec{H}$ is everywhere orthogonal to the velocity vector $\vec{V}$.

(A) Radial Flows. The square of the element of arc length in polar co-ordinate system is given by

$$ds^2 = dr^2 + r^2\, d\theta^2 \tag{64.01}$$

Since the flows are radial, we have

$$\phi = \phi(r) \quad , \quad \psi = \psi(\theta) \tag{64.02}$$

Using (64.02) in (22.02), we get

$$ds^2 = E(\phi')^2\, dr^2 + 2F\, \phi'\psi'\, d\theta\, dr + G(\psi')^2\, d\theta^2 \tag{64.03}$$

Comparison of (64.03) with (64.01) yields

$$E = \frac{1}{(\phi')^2} \; , \quad F = 0 \quad , \quad G = \frac{r^2}{(\psi')^2} \tag{64.04}$$

From (63.07) and (64.04), we have

$$\psi' = \frac{Kr}{\phi'} = A \tag{64.05}$$

where A is an arbitrary constant.

Using (64.04) in (63.05), we find

$$\Omega = \frac{1}{r}\frac{d}{dr}(r\,\phi')$$

$$= \frac{2K}{A} \tag{64.06}$$

where (64.05) has been used.

Similarly, using (64.04) and (64.05) in (63.06), we obtain

$$\omega = 0 \tag{64.07}$$

Equations (62.08) and (62.10) give

$$V = \frac{A}{r} \qquad \text{and} \qquad H = \frac{K}{A}\, r \tag{64.08}$$

where A is an arbitrary constant which can be determined from the boundary conditions.

Substituting (64.06) and (64.07) in (63.01) and (63.02) we get

$$\left.\begin{aligned} \frac{\partial h}{\partial r} &= -\frac{2\mu K^2}{A^2}\, r \\ \text{and} \\ \frac{\partial h}{\partial \theta} &= 0 \end{aligned}\right\} \tag{64.09}$$

which imply that

$$h = -\mu \frac{K^2}{A^2} r^2 + D$$

where D is an arbitrary constant.

Therefore, we have

$$p = -\mu \frac{K^2}{A^2} r^2 - \frac{\rho}{2} \frac{A^2}{r^2} + D$$

(B) Vortex Flows. We investigate the flows for which

$$\psi = \psi(r), \qquad \phi = \phi(\theta) \tag{64.10}$$

where (r,θ) are the polar co-ordinates of any point in the plane of flow.

For this case, we have

$$E = \frac{r^2}{(\phi')^2}, \quad F = 0, \quad G = \frac{1}{(\psi')^2} \tag{64.11}$$

Equation (63.07) gives

$$\phi' = \frac{rK}{\psi'} = A \tag{64.12}$$

where A is an arbitrary constant.

Using (64.11) and (64.12) in (63.05) and (63.06) respectively, we get

$$\left.\begin{aligned} &\Omega = 0 \\ \text{and} \\ &\omega = -\frac{2K}{A} \end{aligned}\right\} \tag{64.13}$$

Substituting (64.11), (64.12) in (62.08) and (62.10), we find

$$\left.\begin{aligned} &V = \frac{K}{A} r \\ \text{and} \\ &H = \frac{A}{r} \end{aligned}\right\} \tag{64.14}$$

Equations (63.01) and (63.02), on using (64.13) become

$$\left.\begin{aligned} &\frac{\partial h}{\partial \theta} = 0 \\ \text{and} \\ &\frac{\partial h}{\partial r} = \rho \frac{2K^2}{A^2} r \end{aligned}\right\} \tag{64.15}$$

Equations (64.15) imply

$$h = \frac{\rho K^2 r^2}{A^2} + D$$

where D is an arbitrary constant.

From the third equation of (61.06) and (64.16), we get

$$p = \frac{\rho K^2 r^2}{A^2} - \frac{\rho}{2} \frac{K^2}{A^2} r^2 + D$$

where the first equation of (64.14) has been used.

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